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On Relations Defined by Generalized Finite Automata

Abstract: A transduction, in the sense of this paper, is a n-ary word relation (which may be a function) describable by a finite directed labeled graph. The notion of n-ary transduction is co-extensive with the Kleenean closure of finite n-ary relations. The 1-ary transductions are exactly the sets recognizable by finite automata. However, for $n > 1$ the relations recognizable by automata constitute a proper subclass of the n-ary transductions. The 2-ary length-preserving transductions constitute the equilibrium (potential) behavior of 1-dimensional, bilateral iterative networks. The immediate consequence relation of various primitive deductive (respectively computational) systems, such as Post normal systems (respectively Turing machines) are examples of transductions. Other riches deductive systems have immediate consequence relations which are not transductions. The closure properties of the class of transductions are studied. The decomposition of transductions into simpler ones is also studied.

1. Introduction

The class of finite automata has correlated with it a class of sets of words which enjoys a rich structure. (See, for example, [RS].) The class is closed under: 1, Boolean operations; 2, *monoid* (semi-group with identity) homomorphisms and their pre-images; 3, word reversal; 4, (binary) concatenation; and 5, (unary) concatenation closure. The class is describable not only by finite automata, but also by sequential machines, finite state grammars, and one-way-motion Turing machines. There are simple intrinsic characterizations of the class. In brief, this class of sets appears to be "natural" and plays a role in mathematical studies other than the theory of finite automata in which it originated. In particular, it may be noted that the one-sided (left to right) orientation present in the notion "finite automaton" is *not* reflected in the associated class of sets (property 3, above).

On the other hand, word relations (other than the unary ones) definable by finite-automaton-like devices have not been widely studied. We consider a class of n-ary relations, $n > 0$, which we call (finite state) transductions, study its structure, and show that it encompasses a wide class of "immediate consequence" relations of formal deductive (particularly, computational) systems [EM]. A subclass of the transductions may be seen to be intimately related to the 1-dimensional, 2-way iterative systems of networks in the sense of [FCH]. The case $n = 2$ is singled out for special study. Inasmuch as we will identify a binary relation R satisfying "$(u, v) \in R \wedge (u, w) \in R \cdot \rightarrow \cdot v = w$" with a function (and we will write, as usual, $R(u) = v$), our considerations will, among other things, yield the "closed under composition" result of [MPS].

While the class of word-to-word mappings associated with sequential machines does possess some nice properties, its usefulness in certain contexts is severely limited by (beside the functionality restriction) the following two properties:

(a) the image of a prefix of a word is a prefix of the image of the word, and
(b) the length of the image of a word is equal to the length of the word.

Condition (b) has been relaxed in several published studies (see, e.g., [SG]), by permitting a machine to "print" a word, possibly null, per input letter. In [MPS] and [RS], however, both restrictions have been removed (and in [RS] the functionality restriction has been removed as well).

The class of transductions we study properly includes the class studied in [MPS] and is the class of relations definable by the multi-tape, 1-way automata of [RS] modified to be nondeterministic with several initial states. The proof of this result is, however, omitted.

We assume throughout that the alphabets Σ are all non-empty finite subsets of some infinite set given once and for all. A relation is a *transduction* if it is a *transduction over* Σ *for some* Σ. Similarly for other notions.

2. Definitions and summary of results

We understand "*finite automaton over a finite alphabet* Σ" as in [RS]. It is inessential but convenient to extend the notion to permit the automaton to have $n \geq 0$ "*input tapes*," each of which has written on it a word (possibly null) in the alphabet of Σ. Each tape is regarded as semi-infinite having written on it, to the right of the word over Σ, an infinite succession of blanks. The automaton starts in a prescribed state, reads simultaneously the first symbol of each tape, changes state, reads simultaneously the second symbol of each tape, changes state, etc., until it reads a blank on each tape. The automaton then stops without changing state and *accepts* the n-tuple of words or not, according to whether or not its final state is a member of a pre-designated set of states. The set of all n-tuples of words accepted by the automaton is the (n-ary) *relation defined by the automaton.* Let Σ^* be the set of all words over Σ including the null (i.e., empty) word Λ. An n-ary relation R, $R \subseteq (\Sigma^*)^n$, is said to be *finite automaton definable* (FAD) if and only if (iff) there exists an automaton such that u, $u \in (\Sigma^*)^n$, is accepted by the automaton when and only when $u \in R$. The automaton in question is described by a *finite* set S of states, a mapping ν from $(S \times (\Sigma \cup \{\beta\})^n) - (S \times \{\beta\}^n)$ into S, an initial state and a subset D of S, where $\beta \notin \Sigma$ and β plays the role of "blank." For example, the ternary length-preserving relation which holds among u, v, w when w is (assuming $\Sigma = \{0, 1, \cdots, p-1\}$) the p-ary representation of the sum of the p-ary numbers u, v is FAD.

By a (finite state) *nondeterministic sequential machine* (NDSM) *over* Σ (*with m inputs and p outputs*),[†] we shall mean an ordered triple (S, ν, s_I) where S is a finite non-empty set (of states), $s_I \in S$ (the initial state), and $\nu \subseteq (S \times (\Sigma^*)^m) \times ((\Sigma^*)^p \times S)$ is *finite*. By a (n-input) *nondeterministic* (*finite*) *automaton over* Σ (NDA) we shall mean an ordered quadruple (S, ν, s_I, D), where S, s_I are as above, $D \subseteq S$ and $\nu \subseteq S \times (\Sigma^*)^n \times S$ is *finite*. [The notion of NDA as it appears in [RS] has been broadened to permit the definition of n-ary relations.] A NDSM is *elementary* if $\nu \subseteq (S \times \Sigma^m) \times (\Sigma^p \times S)$. *In the sequel all* NDSM*'s will be assumed elementary with* $m = 1$ *and* $p = 1$. The sole point in giving a general definition of NDSM is to make clear the distinction we are making between "machine" and "automaton." Informally, at each instant of time the *machine* receives an input and yields an output while the *automaton* responds only after receiving a sequence of inputs by entering or not entering the designated set D of states. This unhappy terminology conforms to common usage. Associated with a given NDSM is a length-preserving relation μ, $\mu \subseteq \Sigma^* \times \Sigma^*$, defined as follows:

† This notion is closely related to the notion "transducer" as used in [NC], p. 33.

(a) $(\Lambda, \Lambda) \in \mu$,

(b) $(\sigma_1\sigma_2 \cdots \sigma_r, \sigma_1'\sigma_2' \cdots \sigma_r') \in \mu$,

if there is a sequence of states $s_1, s_2, \cdots, s_{r+1}$ such that $s_1 = s_I$ and for all i, $1 \geq i \geq r$, $((s_i, \sigma_i), (\sigma_i', s_{i+1})) \in \nu$. If ν is a function, then (S, ν, s_I) is (by definition) a *sequential machine* and μ is a function.* A relation $\mu \subseteq \Sigma^* \times \Sigma^*$ is a *sequential relation* (over Σ) if it is the associated relation of some NDSM over Σ. It may readily be verified, if $\Sigma \subset \Sigma'$ that μ is a sequential relation over Σ iff it is a sequential relation over Σ'. Thus, often we simply say "sequential relation." An n-ary word relation R is said to be *prefix closed* iff (a) *R is length-preserving*[§] (LP), i.e., $(u_1, u_2, \cdots, u_n) \in R \rightarrow l(u_1) = l(u_2) = \cdots = l(u_n)$; and (b) the conjunction of (i) $(u_1, u_2, \cdots, u_n) \in R$, and (ii) v_i is an initial segment (prefix) of u_i, $1 \leq i \leq n$, and (iii) $l(v_1) = l(v_2) = \cdots = l(v_n)$ implies that $(v_1, v_2, \cdots, v_n) \in R$. Again, it may be readily verified that μ, $\mu \subseteq (\Sigma^* \times \Sigma^*)$ *is a sequential relation iff* μ *is nonempty, prefix closed, and* FAD. *If* μ, $\mu \neq \phi$[¶], *is* (i) *prefix closed,* (ii) FAD, *and* (iii) *functional, then there is a sequential machine whose associated function is* μ. The converse follows from the previous sentence (Cf. [CCE], Theorem 7.1). Thus, a sequential relation which is functional is the associated function of a sequential machine and we may call such a function, without ambiguity, a *sequential function.*

If $u = \sigma_1\sigma_2 \cdots \sigma_r \in \Sigma^*$, then the *word reversal* function ρ_Σ takes u into $\sigma_r \cdots \sigma_2\sigma_1$; ρ_Σ is extended to operate on n-tuples of words componentwise and then to operate on n-ary relations memberwise. We shall, when no confusion threatens, drop the subscript from "ρ_Σ". We will show, in Section 7, *Theorem 7.8:*[#] *If* μ *is* LP, FAD *and functional and* $(\Lambda, \Lambda) \in \mu$, *then there are sequential functions* μ_1, μ_2 *such that* $\mu = \rho \circ \mu_2 \circ \rho \circ \mu_1$. We use "$\circ$" for Pierce product**, also called relative product, which, in the case of functions, is simply composition. (If μ_1 is regarded as being effected by a machine which reads and writes on a finite tape in a left-to-right motion, then $\rho \circ \mu_2 \circ \rho$ may be regarded as performed by a similar machine which reads and writes, however, from right to left.) An analogous result holds for sequential relations, but this is easier to demonstrate. On the other hand, *if* μ *is the composite of an even number of word reversals and any finite number of sequential relations, then* μ *is* FAD. An n-ary NDA $\mathfrak{A}$ has associated with it an n-ary relation $T(\mathfrak{A})$ defined as follows. Let $u \in (\Sigma^*)^n$. Then $u \in T(\mathfrak{A})$ iff there exist sequences s_i,

* If ν is functional and (S, ν, s_I) is not elementary, it does not follow that μ is functional.

§ We use $l(u)$ or l_u to denote the length of a word u.

¶ ϕ denotes the empty set.

\# Here and subsequently, theorems, propositions, and other statements are assigned numbers indicating the sections and subsections where they are treated.

** Where R_1, R_2 are binary relations: $(u, v) \in R_2 \circ R_1 \Leftrightarrow \exists w[(u, w) \in R_1 \wedge (w, v) \in R_2]$.

$1 \leq i \leq m+1$, $s_i \in S$, and u_i, $1 \leq i \leq m$, $u_i \in (\Sigma^*)^n$ such that: $s_1 = s_I$, $s_{m+1} \in D$, $(s_i, u_i, s_{i+1}) \in \nu$ and $u = u_1u_2 \cdots u_m$. Here we use juxtaposition to denote concatenation, and concatenation of two n-tuples is performed componentwise. That is, if $u_i = (u_i^1, u_i^2, \cdots, u_i^n)$ and $u_j = (u_j^1, u_j^2, \cdots, u_j^n)$, then $u_iu_j = (u_i^1u_j^1, u_i^2u_j^2, \cdots, u_i^nu_j^n)$ An n-ary relation R, $R \subseteq (\Sigma^*)^n$, is said to be a *transduction* iff there exists an n-ary NDA $\mathfrak{A}$ over Σ such that $R = T(\mathfrak{A})$. An NDA $\mathfrak{A}$ may be graphically described by a labeled, directed graph, in which the nodes correspond to elements of S, each directed edge is *labeled* with (the name of) an element of $(\Sigma^*)^n$, and a certain node is distinguished as s_I and some nodes are distinguished as elements of D. To each *path*† in $\mathfrak{A}$, beginning with s_I and terminating with an element of D, corresponds an element of $(\Sigma^*)^n$, *the label of the path*, obtained by concatenating the labels of the traversed edges in the order traversed. The set of all such elements of $(\Sigma^*)^n$ is $T(\mathfrak{A})$.

If R, S are subsets of $(\Sigma^*)^n$, $RS \overset{\text{def}}{=} \{uv : u \in R, v \in S\}$; $R^* \overset{\text{def}}{=} R^0 \cup R \cup RR \cup RRR \cup \cdots$, where $R^0 = \overbrace{\{(\Lambda, \Lambda, \cdots, \Lambda)\}}^{n}$. We shall show that *the class of n-ary transductions over Σ is exactly the smallest class of subsets of $(\Sigma^*)^n$ which contains the finite n-ary relations and is closed under binary union, (binary) concatenation and (unary) concatenation closure (i.e., $*$)*. This may suggest that a relation is a transduction iff it is FAD. This is indeed the case for $n = 1$; but for $n > 1$, *the* FAD *relations form a proper subclass of the transductions.* As a matter of fact, $\{(0,00)\}^*$ is a transduction which is not FAD; this relation is a homomorphism of $F_\Sigma = (\Sigma^*, \cdot, \Lambda)$, where "$\cdot$" denotes concatenation and $\Sigma = \{0\}$. All *homomorphisms* shall be understood to be of the *free monoid* F_Σ into itself (where Σ is an arbitrary finite set) unless otherwise stated.

We shall also prove *Theorem 4.13*: *The binary transductions are closed under Pierce product (i.e., relative product or composition).* As a matter of fact, *the class of binary transductions is the smallest class of binary relations, closed under composition and conversion, which contains the length-preserving ones and the homomorphisms.* (The *converse* of R is $R^c \overset{\text{def}}{=} \{(u, v) : (v, u) \in R\}$). It will be shown later that the LP *transductions are* FAD.

Among the binary transductions, we shall distinguish a subclass (locally finite) which contains the functional transductions. A relation R is *locally finite* iff for all u, the set $\{v : (u, v) \in R\}$ is finite. It is obvious that if R is a function, it is locally finite.

A relation R is *symmetrically locally finite* iff R and R^c are each locally finite; R is *bounded* iff for some positive integer M, the following holds: for all u, v, if $(u, v) \in R$, then both $l(u) - l(v) \leq M$ and $l(v) - l(u) \leq M$. It is obvious that if R is bounded, it is symmetrically locally finite and that the converse is false. *Theorem 6.1*: *The intersection of the symmetrically locally finite transductions and the* FAD *ones is exactly the class of bounded transductions.* (It is a consequence of this theorem in one direction, that LP transductions are FAD.)

An ordered pair (Λ, u), $u \neq \Lambda$ is said to be *inadmissible.* A locally finite transduction which fails to contain inadmissible pairs is called an *S-transduction.* We present in Section 5 *Theorem 5.1*: *The S-transductions are exactly those of the form $h \circ T$ where T is an* LP *transduction and h is a homomorphism; if the transduction is functional, then T may be chosen functional.*

The accompanying table summarizes the closure properties of the subclasses of binary transductions discussed above. The fifth column deals with symmetric difference and the seventh with concatenation.

As counter-example for the *no*'s of column 4 of Table 1, we have homomorphism restrictions $\varphi_1 = \{(0, 0)\}^* \cdot \{(1,00)\}^*$, $\varphi_2 = \{(0,00)\}^* \cdot \{(1,0)\}^*$. Then $\varphi_1 \cap \varphi_2 = \{(0^n1^n, 0^{3n}) : n \geq 0\}$ and the domain of $\varphi_1 \cap \varphi_2$ is not FAD. Hence, $\varphi_1 \cap \varphi_2$ is *not* a transduction.

As counter-examples for row 5, we have:

Column 6— the initial segment relation is FAD, but the terminal segment relation is not;

Table 1 Closure properties of the subclasses of binary transductions.

	Closed under							
	$\circ$	c	$\cup$	$\cap$	$+$	ρ	$\cdot$	$*$
Transductions	yes	yes	yes	no	no	yes	yes	yes
Locally finite transductions	yes	no	yes	no	no	yes	yes	no
Symmetrically locally finite transductions	yes	yes	yes	no	no	yes	yes	no
S-transductions	yes	no	yes	no	no	yes	yes	yes
FAD transductions	yes	yes	yes	yes	yes	no	no	no
Bounded transductions	yes	yes	yes	yes	yes	yes	yes	no
LP transductions	yes	yes	yes	yes	yes	yes	yes	yes
1-1 functional transductions	yes	yes	no	?	no	yes	no	no

† Strictly speaking, by a path in $\mathfrak{A}$, we mean a sequence of *edges* (s_i, u_i, s_{i+1}), $1 \leq i \leq m$, such that $(s_i, u_i, s_{i+1}) \in \nu$. The path is said to *begin with* s_1 and *terminate with* s_{m+1}; the path *connects* s_1 to s_{m+1}. The path is said to *pass through* s_i, $2 \leq i \leq m$. A path in $\mathfrak{A}$, beginning with s_I and terminating with an element of D will be called *successful.*

Column 7— $\{(0^m, 0^n) : m, n \geq 0\}$ is FAD, as is $\{(1^p, 1^p) : p \geq 0\}$ but their concatenation is not;
Column 8— $\{(0,00)\}$ is FAD, but $\{(0,00)\}^*$ is not.
Counter-examples for the other *no*'s are readily provided.

Let the expression "relation R of rank m" be synonymous with "m-ary relation R". The operations or relations presented in the preceding table were thus all "rank preserving". Considering operations on relations that do, in general, change the rank, we observe that the class of transductions is closed under "generalized composition", "existential quantification", and "Cartesian product" while not closed under "identification of variables". (These terms are defined in Section 8.)

The decomposition results may be summarized as follows. Let R be a binary transduction.

1) $R = R_{lf} \cup R_\infty$, where R_{lf}, R_∞ are the "locally finite" and "locally infinite" parts of R, respectively (Proposition 3.7), both R_{lf}, R_∞ are transductions and $R_{lf} \cap R_\infty = \phi$;
2) $R_{lf} = R_s \cup I'$, where R_s is a S-transduction and I' a (finite) set of inadmissible pairs, $R_s \cap I' = \phi$ (Corollary 3.19, 3.20);
3) $R_s = h \circ T$, where T is an LP transduction and h a homomorphism (Theorem 5.1);
4) $T = \rho(\mu_2) \circ \mu_1$, where μ_1, μ_2 are sequential relations; (Proposition 7.4).

Thus, $R = h \circ \rho(\mu_2) \circ \mu_1 \cup I' \cup R_\infty$.

In the case that R is functional, $R_\infty = \phi$, I' contains at most one element and μ_1, μ_2 may be chosen functional.

The (potential) equilibrium behavior of a (one-dimensional) iterative system [FCH] is an LP transduction. The "immediate consequence relations" of combinatorial systems in the sense of [MD], (which include the normal systems of Post), are transductions as well as the "atomic step functions" of Turing machines and Markov algorithms. Among arithmetic relations p-ary addition, $p \geq 2$, is a transduction.

All the preceding relations are FAD and bounded. The concatenation relation serves as an example of a non-FAD transduction.

Among the examples of relations that are not transductions, the multiplication relations, both unary and binary, are discussed. These examples appear in Section 9.

3. The Kleenean closure of finite relations

It is well known (see, e.g., [RS]) that for $n = 1$ the properties of subsets R of $(\Sigma^*)^n$ defined in any of the following three ways are coextensive:

1) R is obtainable from finite subsets of $(\Sigma^*)^n$ by a finite number of applications of $\cup$, $\cdot$, $*$;
2) $R = T(\mathfrak{A})$ for some NDA $\mathfrak{A}$;
3) $R = T(\mathfrak{A})$ for some $\mathfrak{A}$ satisfying: for each $u \in (\Sigma^*)^n$, there is exactly one path in $\mathfrak{A}$ starting in s_I and with label u. It is shown in this section that the equivalence between (1) and (2) persists for arbitrary n, thus providing an alternative definition of "transduction".

The persistence of equivalence between (1) and (2) reflects the fact that in many arguments the notions (1) and (2) are interchangeable. Notion (1) has the advantage of being more algebraic and thus lending itself to sharper proofs, while (2) has the advantage of being more intuitive.†

The equivalence between (2) and (3), however, fails for all $n > 1$ because: if (2) and (3) were equivalent, then the class of sets satisfying (2) would be closed under complementation.

The class of n-ary transductions, $n > 1$, is not closed under set subtraction; it is shown in Corollary 3.17, however, that subtraction of finite sets from transductions yields transductions. The main result of this section for later use is Corollary 3.20 which is the main tool in the proof of the Theorem 5.1.

Definition 3.1: Let $\mathfrak{E}$ be a family of subsets of $(\Sigma^*)^n$. The *Kleenean closure* $\mathfrak{K}(\mathfrak{E})$ of $\mathfrak{E}$ is the smallest class of subsets of $(\Sigma^*)^n$ such that $\mathfrak{E} \subseteq \mathfrak{K}(\mathfrak{E})$ and $\mathfrak{K}(\mathfrak{E})$ is closed under (binary) union and concatenation and the (unary) concatenation closure $*$. $\mathfrak{E}$ is referred to as the collection of *atoms* of $\mathfrak{K}(\mathfrak{E})$.

Definition 3.2: Let Λ denote, *ambiguously*, the n-tuple $\underbrace{(\Lambda, \cdots, \Lambda)}_{n} \in (\Sigma^*)^n$ for any n, depending on context to remove the ambiguity. Similarly, let $\{\Lambda\}$ denote $\underbrace{\{(\Lambda, \cdots, \Lambda)\}}_{n}$ for any n.

$\mathfrak{K}(\mathfrak{E})$ may be obtained "from the inside" as follows. Define $\mathfrak{E}_0 \overset{\text{def}}{=} \mathfrak{E}$, and for all $i \geq 0$:

$$\mathfrak{E}_{i+1} \overset{\text{def}}{=} \{R : \exists R_1, R_2 \in \mathfrak{E}_i \; [R = R_1 \cup R_2 \vee R = R_1 R_2 \vee R = R_1^*]\}$$

Clearly, $\mathfrak{E}_{i+1} \supseteq \mathfrak{E}_i$ and $\mathfrak{K}(\mathfrak{E}) = \bigcup_{i=0}^{\infty} \mathfrak{E}_i$. To prove that $\mathfrak{K}(\mathfrak{E})$ has some property ρ, one may use the principle of mathematical induction: it is necessary and sufficient to prove $\mathfrak{E}_0$ has property ρ and that given any natural number i if $\mathfrak{E}_i$ has the property ρ, then so does $\mathfrak{E}_{i+1}$. Such a proof we subsequently refer to as "by induction".

Definition 3.3: In preparation for the next proposition, we introduce the following notion. A NDA $\mathfrak{A} = (S, \nu, s_I, D)$ is *simple* iff:

(1) $D = \{s_T\}$ for some $s_T \in S$
(2) $s_I \neq s_T$
(3) $\forall s \forall u \in (\Sigma^*)^n [(s, u, s_I) \notin \nu]$
(4) $\forall s \forall u \in (\Sigma^*)^n [(s_T, u, s) \notin \nu]$

† In fact, there is a valid analog of Proposition 3.5 in which the system consisting of n-tuples of words under concatenation is replaced by an arbitrary semi-group.

i.e., D consists of exactly one terminal state s_T, $s_I \neq s_T$ and all edges "attached" to s_I (resp. s_T) are directed away from (resp. toward) it. The nodes s_I, s_T are called "extremal" nodes.

We leave to the reader to verify the following:

Lemma 3.4: Given any transduction R, there exists a simple NDA such that $R = T(\mathcal{A})$.

Proposition 3.5: The class of n-ary transductions over Σ is precisely $\mathfrak{R}(\mathfrak{F})$, where $\mathfrak{F}$ is the class of all finite subsets* of $(\Sigma^*)^n$.

Proof: The class of n-ary transductions over Σ is closed under (a) union, (b) concatenation, and (c) concatenation closure because, given *simple* automata $\mathcal{A}_i$, $i = 1, 2$:

(a) Identifying initial and terminal nodes of the two automata respectively produces an automaton $\mathcal{A}$ such that $T(\mathcal{A}) = T(\mathcal{A}_1) \cup T(\mathcal{A}_2)$.

(b) Identifying the terminal node of the first automaton with the initial node of the second produces an automaton $\mathcal{A}$ such that $T(\mathcal{A}) = T(\mathcal{A}_1) \cdot T(\mathcal{A}_2)$.

(c) Identifying the initial and terminal nodes of $\mathcal{A}_1$ produces an automaton $\mathcal{A}$ (not simple) such that $T(\mathcal{A}) = (T(\mathcal{A}_1))^*$. Since every $F \in \mathfrak{F}$ is a transduction, every set in $\mathfrak{R}(\mathfrak{F})$ is a transduction.

To show that every transduction is in $\mathfrak{R}(\mathfrak{F})$, we employ an induction on the number n of nonextremal nodes of a simple automaton that defines the given transduction.

Suppose $R = T(\mathcal{A})$, where simple automaton $\mathcal{A}$ has $n > 0$ nonextremal nodes, and that for all simple automata $\mathcal{A}'$ with fewer nonextremal nodes $T(\mathcal{A}')$ is in $\mathfrak{R}(\mathfrak{F})$. (It is clear that if $n = 0$, $T(\mathcal{A}) \in \mathfrak{R}(\mathfrak{F})$.) Let s_M be a state of $\mathcal{A}$ different from s_I, s_T. Let:

$P_{I,T} \overset{\text{def}}{=}$ the set of all successful paths in $\mathcal{A}$;

$P_M \overset{\text{def}}{=}$ the set of all successful paths in $\mathcal{A}$, which pass through s_M;

$P_{\bar{M}} \overset{\text{def}}{=}$ the set of all successful paths in $\mathcal{A}$, which do *not* pass through s_M.

Clearly, $P_{I,T} = P_M \cup P_{\bar{M}}$ and if $\lambda(P)$ is the set of labels of paths¶ in P, then:

$$T(\mathcal{A}) = \lambda(P_{I,T}) = \lambda(P_M) \cup \lambda(P_{\bar{M}}).$$

If we delete s_M from $\mathcal{A}$ and restrict ν accordingly, we obtain a simple automaton $\mathcal{A}_M^\lambda$ such that $T(\mathcal{A}_{\bar{M}}) = \lambda(P_{\bar{M}})$ and hence, by the inductive assumption, $\lambda(P_{\bar{M}}) \in \mathfrak{R}(\mathfrak{F})$. We now show $\lambda(P_M)$ is in $\mathfrak{R}(\mathfrak{F})$ as well.

P_M decomposes as follows: $P_M = P_{I,M} \cdot P_{M,M} \cdot P_{M,T}$, where

$P_{I,M} \overset{\text{def}}{=}$ the set of all paths in $\mathcal{A}$ from s_I to s_M which do *not* pass through s_M;

$P_{M,M} \overset{\text{def}}{=}$ the set of all paths in $\mathcal{A}$ from s_M to s_M;

$P_{M,T} \overset{\text{def}}{=}$ the set of all paths in $\mathcal{A}$ from s_M to s_T, which do *not* pass through s_M.

Let $\mathcal{A}_{I,M}$ be the automaton obtained from $\mathcal{A}$ by deleting s_T and all edges of $\mathcal{A}$ "attached" to s_T as well as the edges "attached" to s_M that are directed away from s_M. The initial state of $\mathcal{A}_{I,M}$ is s_I and its terminal state is s_M. Then, $T(\mathcal{A}_{I,M}) = \lambda(P_{I,M})$. Inasmuch as $\mathcal{A}_{I,M}$ is simple with $n - 1$ nonextremal nodes, by the inductive assumption $T(\mathcal{A}_{I,M}) \in \mathfrak{R}(\mathfrak{F})$. In a similar way, it is shown that $\lambda(P_{M,T}) \in \mathfrak{R}(\mathfrak{F})$.

Let $\mathcal{A}_{M,M}$ be the automaton derived from $\mathcal{A}$ by deleting s_I, s_T and restricting ν accordingly and taking s_M as both the initial and terminal state of $\mathcal{A}_{M,M}$. Then, $T(\mathcal{A}_{M,M}) = \lambda(P_{M,M})$. Further, let $\mathcal{A}'$ be the automaton derived from $\mathcal{A}_{M,M}$ by "splitting" the node s_M into the two distinct nodes s_I', s_T' (the initial and terminal nodes of $\mathcal{A}'$ respectively) in such a manner that $T(\mathcal{A}_{M,M}) = (T(\mathcal{A}'))^*$. Then, $\mathcal{A}'$ is simple with $n - 1$ nonextremal nodes and, by the inductive assumption, $T(\mathcal{A}') \in \mathfrak{R}(\mathfrak{F})$. Then, $T(\mathcal{A}_{M,M}) = \lambda(P_{M,M}) \in \mathfrak{R}(\mathfrak{F})$. Since $\lambda(P_M) = \lambda(P_{I,M}) \cdot \lambda(P_{M,M}) \cdot \lambda(P_{M,T})$, $\lambda(P_M) \in \mathfrak{R}(\mathfrak{F})$ and finally $\lambda(P_{I,T}) = T(\mathcal{A}) \in \mathfrak{R}(\mathfrak{F})$, which completes the proof. □

Corollary 3.6: The class of locally finite (resp. bounded) binary transductions over Σ is precisely the class of relations obtained from $\mathfrak{F}$ by a finite number of applications of $\cup$, $\cdot$, $*$ where $*$ is restricted to apply only to relations without inadmissible pairs (resp. to LP relations).

This corollary may be proved by induction, utilizing the following two observations. Given nonempty binary relations S, T:

(a) $S \cup T$, ST are locally finite (resp. bounded) iff both S, T are locally finite (resp. bounded)

(b) S^* is locally finite (resp. bounded) iff S is locally finite and contains no inadmissible pairs (resp. S is LP).

Proposition 3.7: Given a (binary) relation $R \subseteq (\Sigma^*)^2$. Let $\mathrm{dom}_\infty R$ be defined as the set of sequences u such that $\{v\colon (u, v) \in R\}$ is infinite. Let $R_\infty \overset{\text{def}}{=} \mathrm{dom}_\infty R \restriction R$, $R_{lf} \overset{\text{def}}{=} (\Sigma^* - \mathrm{dom}_\infty R) \restriction R$. Call R_∞, R_{lf} the "locally infinite" and "locally finite" parts of R respectively (disjoint, possibly empty).† We claim R is a transduction iff R_∞, R_{lf} are both transductions.

Proof: Since $R = R_\infty \cup R$ one need show only that if R is a transduction so are R_∞, R. It is in fact sufficient, using Corollary 4.10, to show that $\mathrm{dom}_\infty R$ is FAD. Proof is by induction.

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* The generating set $\mathfrak{F}$ of $\mathfrak{R}(\mathfrak{F})$ may be reduced to a finite subset $\mathcal{S}$. Consider binary transductions over the alphabet $\{0, 1\}$. Let $\mathcal{S} \overset{\text{def}}{=} \{\phi, \{(\Lambda, 0)\}, \{(\Lambda, 1)\}, \{(0, \Lambda)\}, \{(1, \Lambda)\}\}$. Then $\mathcal{S}$ is finite and $\mathfrak{R}(\mathcal{S}) = \mathfrak{R}(\mathfrak{F})$.

¶ λ is a function that associates with each path in $\mathcal{A}$ the label of that path.

† 1) If $R \subseteq A \times B$, dom $R = \{u : (u, v) \in R\}$, ran $R = \{v : (u, v) \in R\}$

2) Let $R \subseteq A \times B$, Then, by $C \restriction R$ is meant the restriction of R to C, i.e., $C \restriction R = \{(a, b) : (a, b) \in R \wedge a \in C\}$

If $R \in \mathfrak{F}_0 = \mathfrak{F}$, $\mathrm{dom}_\infty R = \phi$, thus FAD. Let $S, T \in \mathfrak{F}_i$ and consider:

(a) $R = S \cup T$ or $R = ST$.
Then, $\mathrm{dom}_\infty R = \mathrm{dom}_\infty S \cup \mathrm{dom}_\infty T$
$\mathrm{dom}_\infty R = \mathrm{dom}_\infty S \; \mathrm{dom}\, T \cup \mathrm{dom}\, S \; \mathrm{dom}_\infty T$,
respectively. By induction assumption, $\mathrm{dom}_\infty S$, $\mathrm{dom}_\infty T$ are FAD, and from 8.2 dom S, dom T are FAD, hence $\mathrm{dom}_\infty R$ is FAD.

(b) $R = S^*$.
Distinguish the two cases:
(b_1) S contains an inadmissible pair $(\Lambda, u) \neq \Lambda$. Then $\mathrm{dom}_\infty R = \mathrm{dom}\, R$;
(b_2) S contains no inadmissible pair. Then, $\mathrm{dom}_\infty R = \mathrm{dom}\, S^* \; \mathrm{dom}_\infty S \; \mathrm{dom}\, S^*$.

By inductive assumption $\mathrm{dom}_\infty S$ is FAD and dom R is FAD by 8.2, hence $\mathrm{dom}_\infty R$ is FAD. Thus, given the proposition holds for all $R \in \mathfrak{F}_i$ it holds for all $R \in \mathfrak{F}_{i+1}$, which completes the proof. □

The "locally finite part" of a binary transduction R may still contain a (finite) set of inadmissible pairs. In the remainder of this section we establish that taking away these inadmissible pairs leaves an S-transduction R_s and, more important, denoting by $\mathfrak{F}^a$ the collection of finite subsets of $(\Sigma^*)^2$ whose members are all admissible pairs, $R_s \in \mathfrak{K}(\mathfrak{F}^a)$.

To continue, we need to introduce a new concatenation closure † and the associated Kleenean closure $\mathfrak{L}$.

Definition 3.8: The (unary) operation † on subsets $R \subseteq (\Sigma^*)^n$ is defined by:

$$R^\dagger = R \cup RR \cup RRR \cup \cdots$$
$$= RR^* = R^* - \{\Lambda\}$$

If $\mathfrak{E}$ is a family of subsets of $(\Sigma^*)^n$, then $\mathfrak{L}(\mathfrak{E})$ is the smallest class of subsets of $(\Sigma^*)^n$ such that $\mathfrak{E} \subseteq \mathcal{L}(\mathfrak{E})$ and $\mathfrak{L}(\mathfrak{E})$ is closed under (binary) union and concatenation and the (unary) operation †.

The collections $\mathfrak{E}_i^\dagger$, $0 \leq i$ are defined analogously to the case of Kleenean closure and $\mathfrak{L}(\mathfrak{E}) = \bigcup_{i=0}^{\infty} \mathfrak{E}_i^\dagger$ so that proofs may be carried out by induction.

Definition 3.9: Given a family $\mathfrak{E}$ of subsets of $(\Sigma^*)^n$ and a subset $B \subseteq (\Sigma^*)^n$, define the family:

$$\mathfrak{E}_B \overset{\text{def}}{=} \{A - B : A \in \mathfrak{E}\}.$$

In particular, $\mathfrak{E}_{\{\Lambda\}} = \{A - \{\Lambda\} : A \in \mathfrak{E}\}$ and for compactness, we write $\mathfrak{E}_\Lambda$ for $\mathfrak{E}_{\{\Lambda\}}$.

The following properties follow immediately from these definitions.

(1) $\mathfrak{K}(\mathfrak{K}(\mathfrak{E})) = \mathfrak{K}(\mathfrak{E})$
$\mathfrak{L}(\mathfrak{L}(\mathfrak{E})) = \mathfrak{L}(\mathfrak{E})$
$(\mathfrak{E}_\Lambda)_\Lambda = \mathfrak{E}_\Lambda$

(2) $\mathfrak{E} \subseteq \mathfrak{F} \Rightarrow \mathfrak{K}(\mathfrak{E}) \subseteq \mathfrak{K}(\mathfrak{F})$
$\wedge\ \mathfrak{L}(\mathfrak{E}) \subseteq \mathfrak{L}(\mathfrak{F})$
$\wedge\ \mathfrak{E}_\Lambda \subseteq \mathfrak{F}_\Lambda$

(3) $\mathfrak{L}(\mathfrak{E}) \subseteq \mathfrak{K}(\mathfrak{E})$

Lemma 3.10: $(\mathfrak{K}(\mathfrak{E}))_\Lambda \subseteq \mathfrak{K}(\mathfrak{E}_\Lambda)$.

Proof: By induction. If $R \in \mathfrak{E} = \mathfrak{E}_0$, then $R - \{\Lambda\} \in \mathfrak{E}_\Lambda = (\mathfrak{E}_\Lambda)_0$. Let $R_1, R_2 \in \mathfrak{E}_i$ and consider the cases:

(a) $R = R_1 \cup R_2$. Then:
$R - \{\Lambda\} = (R_1 - \{\Lambda\}) \cup (R_2 - \{\Lambda\})$

(b) $R = R_1 R_2$. Then:

$$R - \{\Lambda\} = \begin{cases} (R_1 - \{\Lambda\})(R_2 - \{\Lambda\}) \\ \qquad \text{if} \quad \Lambda \notin R_1 \wedge \Lambda \notin R_2 \\ (R_1 - \{\Lambda\})(R_2 - \{\Lambda\}) \\ \cup\ (R_1 - \{\Lambda\}) \\ \qquad \text{if} \quad \Lambda \notin R_1 \wedge \Lambda \in R_2 \\ (R_1 - \{\Lambda\})(R_2 - \{\Lambda\}) \\ \cup\ (R_2 - \{\Lambda\}) \\ \qquad \text{if} \quad \Lambda \in R_1 \wedge \Lambda \notin R_2 \\ (R_1 - \{\Lambda\})(R_2 - \{\Lambda\}) \\ \cup\ (R_1 - \{\Lambda\}) \cup (R_2 - \{\Lambda\}) \\ \qquad \text{if} \quad \Lambda \in R_1 \wedge \Lambda \in R_2 \end{cases}$$

(c) $R = (R_1)^* = (R_1 - \{\Lambda\})^*$. Then:
$$R - \{\Lambda\} = (R_1 - \{\Lambda\})(R_1 - \{\Lambda\})^*.$$

By inductive assumption, $R_j - \{\Lambda\} \in \mathfrak{K}(\mathfrak{E}_\Lambda)$ for $j = 1, 2$; hence, $R - \{\Lambda\} \in \mathfrak{K}(\mathfrak{E}_\Lambda)$ in all three cases. Thus, if the lemma holds for all $R \in \mathfrak{E}_i$, it holds for all $R \in \mathfrak{E}_{i+1}$, which completes the proof. □

Lemma 3.11: If $\mathfrak{E}_\Lambda = \mathfrak{E}$, then $(\mathfrak{K}(\mathfrak{E}))_\Lambda = \mathfrak{L}(\mathfrak{E})$.

Proof: In general, $\mathfrak{L}(\mathfrak{E}) \subseteq \mathfrak{K}(\mathfrak{E})$ so that $(\mathfrak{L}(\mathfrak{E}))_\Lambda \subseteq (\mathfrak{K}(\mathfrak{E}))_\Lambda$. If $\mathfrak{E}_\Lambda = \mathfrak{E}$, then $(\mathfrak{L}(\mathfrak{E}))_\Lambda = \mathfrak{L}(\mathfrak{E})$ so that:

$$\mathfrak{L}(\mathfrak{E}) \subseteq (\mathfrak{K}(\mathfrak{E}))_\Lambda$$

To establish $(\mathfrak{K}(\mathfrak{E}))_\Lambda \subseteq \mathfrak{L}(\mathfrak{E})$, we employ induction. If $R \in \mathfrak{E} = \mathfrak{E}_0$, then $R - \{\Lambda\} \in \mathfrak{E}_\Lambda = \mathfrak{E} = \mathfrak{E}_0^\dagger$. Let $R_1, R_2 \in \mathfrak{E}_i$; the cases

(a) $R = R_1 \cup R_2$, and

(b) $R = R_1 R_2$, are dealt with as in the preceding lemma. In the third case,

(c) $R = (R_1)^* = (R_1 - \{\Lambda\})^*$. Then
$$R - \{\Lambda\} = (R_1 - \{\Lambda\})^\dagger.$$

By inductive assumption, $R_j - \{\Lambda\} \in \mathfrak{L}(\mathfrak{E})$ for $j = 1, 2$; hence, $R - \{\Lambda\} \in \mathfrak{L}(\mathfrak{E})$ for all three cases. Thus, if the claim holds for all $R \in \mathfrak{E}_i$, it holds for all $R \in \mathfrak{E}_{i+1}$ which completes the proof. □

Corollary 3.12:

(a) $(\mathfrak{R}(\mathfrak{E}_\Lambda))_\Lambda = \mathfrak{L}(\mathfrak{E}_\Lambda)$,

(b) $(\mathfrak{R}(\mathfrak{E}))_\Lambda \subseteq \mathfrak{L}(\mathfrak{E}_\Lambda)$.

Proof: Since $(\mathfrak{E}_\Lambda)_\Lambda = \mathfrak{E}_\Lambda$, the lemma yields (a) directly. From the preceding lemma:

$$(\mathfrak{R}(\mathfrak{E}))_\Lambda \subseteq (\mathfrak{R}(\mathfrak{E}_\Lambda))_\Lambda.$$

Using (a), one obtains (b). □

Proposition 3.13: If $\mathfrak{E}_\Lambda \subseteq \mathfrak{R}(\mathfrak{E})$, then $(\mathfrak{R}(\mathfrak{E}))_\Lambda = \mathfrak{L}(\mathfrak{E}_\Lambda) \subseteq \mathfrak{R}(\mathfrak{E})$.

Proof: Note that:

$$\mathfrak{E}_\Lambda \subseteq \mathfrak{R}(\mathfrak{E}) \Rightarrow \mathfrak{R}(\mathfrak{E}_\Lambda) \subseteq \mathfrak{R}\mathfrak{R}(\mathfrak{E}) = \mathfrak{R}(\mathfrak{E})$$

so that $\mathfrak{L}(\mathfrak{E}_\Lambda) \subseteq \mathfrak{R}(\mathfrak{E}_\Lambda) \subseteq \mathfrak{R}(\mathfrak{E})$.
By virtue of the corollary to the previous lemma, part (b), we need to establish only $\mathfrak{L}(\mathfrak{E}_\Lambda) \subseteq (\mathfrak{R}(\mathfrak{E}))_\Lambda$. From part (a) of that corollary $\mathfrak{L}(\mathfrak{E}_\Lambda) = (\mathfrak{R}(\mathfrak{E}_\Lambda))_\Lambda$. On the other hand:

$$\mathfrak{E}_\Lambda \subseteq \mathfrak{R}(\mathfrak{E}) \Rightarrow (\mathfrak{R}(\mathfrak{E}_\Lambda))_\Lambda \subseteq (\mathfrak{R}\mathfrak{R}(\mathfrak{E}))_\Lambda = (\mathfrak{R}(\mathfrak{E}))_\Lambda$$

which concludes the proof. □

Proposition 3.14: For all $R \in \mathfrak{L}(\mathfrak{E})$, there exist A_i, $A_j \in \mathfrak{E}$, $U_i \in \mathfrak{L}(\mathfrak{E})$, $1 \le i \le k$, $1 \le j \le r$ such that

$$R = \bigcup_{j=1}^{r} A_j \cup \bigcup_{i=1}^{k} A_i U_i$$

where one of the finite unions $\bigcup_{j=1}^{r} A_j, \bigcup_{i=1}^{k} A_i U_i$ may be vacuous.

Proof: By induction. If $R \in \mathfrak{E} = \mathfrak{E}_0$, then $R = A$, which is of the given form. Suppose R_1, $R_2 \in \mathfrak{E}_i$ are of the given form, then simply by the distributivity of concatenation over union $R_1 \cup R_2$, R_1R_2 and $(R_1)^\dagger = R_1 \cup R_1(R_1)^\dagger$ are all of the given form. Thus, if all $R \in \mathfrak{E}_i$ is of the given form, then all $R \in \mathfrak{E}_{i+1}$ are of the given form, which completes the proof. □

We wish to ask whether, given a transduction R, the collection R' of those n-tuples of R whose lengths exceed a nonnegative integer m is a transduction. (The length of an n-tuple is the maximum among the lengths of its components.) Further, if this R' is a transduction, we ask whether it may be specified in terms only of n-tuples whose lengths exceed m. The answer to both questions is *yes*, as a consequence of the following proposition.

Proposition 3.15: Let $\mathfrak{F}$ denote the collection of finite subsets of $(\Sigma^*)^n$. Given a nonnegative m, define $M \in \mathfrak{F}$, $\mathfrak{F}^> \subseteq \mathfrak{F}$ by:

$$(u_1, \cdots u_n) \in M \Leftrightarrow l_{u_1}, \cdots, l_{u_n} \le m$$

$$F \in \mathfrak{F}^> \Leftrightarrow F \in \mathfrak{F} \wedge F \cap M = \phi.$$

Then,

$$R \in \mathfrak{L}(\mathfrak{F}_\Lambda) \Rightarrow R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$$

where $\bar{M} = (\Sigma^*)^n - M$.

Proof: By induction. If $R \in \mathfrak{F}_\Lambda = (\mathfrak{F}_\Lambda)_0$, then $R \cap \bar{M} \in \mathfrak{F}^>$. Let $S, T \in (\mathfrak{F}_\Lambda)_i$ and consider the cases:

(a) $R = S \cup T$. Then,
$R \cap \bar{M} = (S \cap \bar{M}) \cup (T \cap \bar{M})$.
By the inductive assumption $S \cap \bar{M}, T \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$, hence $R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$.

(b) $R = ST$. Then,

$$R = (S \cap \bar{M})(T \cap \bar{M}) \cup (S \cap M)(T \cap \bar{M}) \cup (S \cap \bar{M})(T \cap M) \cup (S \cap M)(T \cap M).$$

By the inductive assumption, $S \cap \bar{M}, T \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$ so that the term $(S \cap \bar{M})(T \cap \bar{M}) \in \mathfrak{L}(\mathfrak{F}^>)$. Consider next the term $(S \cap M)(T \cap \bar{M})$. Using both the inductive assumption and Proposition 3.14, there exist $F_i, F_j \in \mathfrak{F}^>$ and $U_i \in \mathfrak{L}(\mathfrak{F}^7)$ such that:

$$T \cap \bar{M} = \bigcup_{j=1}^{r} F_j \cup \bigcup_{i=1}^{k} F_i U_i$$

and

$$(S \cap M)(T \cap \bar{M}) = \bigcup_{j=1}^{r} (S \cap M) F_j \cup \bigcup_{i=1}^{k} (S \cap M) F_i U_i.$$

Since for any i:

$$S \cap M \in \mathfrak{F} \wedge F_i \in \mathfrak{F}^> \Rightarrow (S \cap M) F_i \in \mathfrak{F}^>$$

one concludes $(S \cap M)(T \cap \bar{M}) \in \mathfrak{L}(\mathfrak{F}^>)$. A symmetric proof shows $(S \cap \bar{M})(T \cap M) \in \mathfrak{L}(\mathfrak{F}^>)$. Since $(S \cap M)(T \cap M) \in \mathfrak{F}_\Lambda$, $(S \cap M)(T \cap M) \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$, hence $R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$. We remark that from the preceding (a), (b) it is possible to conclude that if R is a polynomial in union, concatenation and sets $R_1, \cdots R_\rho$ such that for $1 \le j \le \rho$, $R_j \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$, then $R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$.

(c) $R = S^\dagger$. Then,

$$R = [S \cup SS \cup \cdots \cup \overbrace{(S \cdots S)}^{nm}] \cup \overbrace{(S \cdots S)}^{nm+1}{}^\dagger \cup [S \cup SS \cup \cdots \cup \overbrace{(S \cdots S)}^{nm}]\overbrace{(S \cdots S)}^{nm+1}{}^\dagger.$$

Denote $\overbrace{S \cdots S}^{nm+1}$ by S^{nm+1}. Then, we claim that using the inductive assumption:

$$S^{nm+1} \cap M = \phi \Rightarrow (S^{nm+1})^\dagger \in \mathfrak{L}(\mathfrak{F}^>) \Rightarrow R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>).$$

Indeed, S^{nm+1} is a polynomial in concatenation and S, and by inductive assumption $S \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$. Hence, $S^{nm+1} \cap \bar{M} = S^{nm+1} \in \mathfrak{L}(\mathfrak{F}^>)$ and $(S^{nm+1})^\dagger \in \mathfrak{L}(\mathfrak{F}^>)$. The expansion of R shows R as a polynomial in union, concatenation and sets S, $(S^{nm+1})^\dagger$ such that:

$$S \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>),$$

$$(S^{nm+1})^\dagger \cap \bar{M} = (S^{nm+1})^\dagger \in \mathfrak{L}(\mathfrak{F}^>);$$

hence, $R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$. Thus, we have reduced the proof to establishing that $S^{nm+1} \cap M = \phi$, i.e., all n-tuples of S^{nm+1} exceed m in length.

Each $s_1 \cdots s_{nm+1} \in S^{nm+1}$ is a product of $nm + 1$ factors $s_i \in S$. Since $S \in \mathfrak{L}(\mathfrak{F}_\Lambda)$, the length of each s_i is at least one: each s_i contributes to the length of at least one of the n components of the product, so that finally there will exist a component of the product of length exceeding m. But then the length of $s_1 \cdots s_{nm+1}$ exceeds m, and $S^{nm+1} \cap M = \phi$.

Thus, if all $R \in (\mathfrak{F}_\Lambda)_i$ obey the lemma, all $R \in (\mathfrak{F}_\Lambda)_{i+1}$ obey it, which completes the proof. □

Corollary 3.16: $R \in \mathfrak{R}(\mathfrak{F}) \Rightarrow R \cap \bar{M} \in \mathfrak{R}(\mathfrak{F}^>)$.

Proof: Note that $\mathfrak{F}_\Lambda \subseteq \mathfrak{F} \Rightarrow (\mathfrak{R}(\mathfrak{F}))_\Lambda = \mathfrak{L}(\mathfrak{F}_\Lambda)$, (Proposition 3.13). Then:

$$R \in \mathfrak{R}(\mathfrak{F}) \Rightarrow R - \{\Lambda\} \in \mathfrak{L}(\mathfrak{F}_\Lambda).$$

Using the proposition:

$$R - \{\Lambda\} \in \mathfrak{L}(\mathfrak{F}_\Lambda) \Rightarrow (R - \{\Lambda\}) \cap \bar{M} = R \cap \bar{M} \in \mathfrak{L}(\mathfrak{F}^>)$$

Finally, since $(\mathfrak{F}^>)_\Lambda = \mathfrak{F}^> \Rightarrow \mathfrak{L}(\mathfrak{F}^>) \subseteq \mathfrak{R}(\mathfrak{F}^>)$, (Proposition 3.13), $R \cap \bar{M} \in \mathfrak{R}(\mathfrak{F}^>)$. □

Corollary 3.17: $R \in \mathfrak{R}(\mathfrak{F})_\wedge F \in \mathfrak{F} \Rightarrow R - F \in \mathfrak{R}(\mathfrak{F})$.

Proof: Define m as the maximal length of a n-tuple in F:

$$m \stackrel{\text{def}}{=} \max \{l_v : (u_1, \cdots u_n) \in F \wedge \exists_1^n j(u_j = v)\}$$

Then, $F \subseteq M$ and $R - F = (R \cap M - F) \cup R \cap \bar{M}$. Now, $R \cap M - F \in \mathfrak{F}$ and by virtue of the preceding corollary, $R \cap \bar{M} \in \mathfrak{R}(\mathfrak{F}^>) \subseteq \mathfrak{R}(\mathfrak{F})$. Hence, $R - F \in \mathfrak{R}(\mathfrak{F})$. □

In what follows, we restrict attention once more to binary relations. From the definitions, we know that a locally finite transduction R contains a finite number of inadmissible pairs, and from the preceding proposition, it follows that a locally finite transduction R is the union of a finite set R_i of inadmissible pairs and an S-transduction R_s. We wish to ask whether the S-transduction R_s may be specified in terms only of admissible pairs. The answer is *yes*, as a consequence of the following proposition.

Proposition 3.18: Let $\mathfrak{F}$ denote the collection of finite subsets of $(\Sigma^*)^2$. Define $I \subseteq (\Sigma^*)^2$ as the set of inadmissible pairs and $\mathfrak{F}^a \subseteq \mathfrak{F}$ as the collection of those finite relations which contain only admissible pairs:

$$I = \{(\Lambda, v) : v \in \Sigma^* \wedge v \neq \Lambda\},$$

$$F \in \mathfrak{F}^a \Leftrightarrow F \in \mathfrak{F} \wedge F \cap I = \phi.$$

Then:

$$R \in \mathfrak{L}(\mathfrak{F}_\Lambda) \wedge R \textit{ is locally finite} \Rightarrow R - I \in \mathfrak{L}(\mathfrak{F}_\Lambda^a).$$

Proof: By induction. For $R \in \mathfrak{F}_\Lambda = (\mathfrak{F}_\Lambda)_0$, $R - I \in \mathfrak{F}_\Lambda^a$. Let $S, T \in (\mathfrak{F}_\Lambda)_i$ and consider the cases:

(a) $R = S \cup T$,

(b) $R = ST$, R nonempty.

In both cases, since R is locally finite, both S and T are locally finite. In particular, $S \cap I$, $T \cap I \in \mathfrak{F}$. With this in mind, using the same proof as in the corresponding cases of the preceding proposition, we conclude $R - I \in \mathfrak{L}(\mathfrak{F}_\Lambda^a)$.

(c) $R = S^\dagger$.

Since $S^\dagger$ is locally finite, S may not contain inadmissibles. By the inductive assumption, $S - I = S \in \mathfrak{L}(\mathfrak{F}_\Lambda^a)$, hence $R \in \mathfrak{L}(\mathfrak{F}_\Lambda^a)$.

Thus, if all $R \in (\mathfrak{F}_\Lambda)_i$ obey the proposition, all $R \in (\mathfrak{F}_\Lambda)_{i+1}$ obey it, which completes the induction. □

Corollary 3.19: $R \in \mathfrak{R}(\mathfrak{F}) \wedge R$ is locally finite $\Rightarrow R - I \in \mathfrak{R}(\mathfrak{F}^a)$.

Proof: Analogous to that of Corollary 3.16.

Corollary 3.20: The class of S-transductions is precisely the class $\mathfrak{R}(\mathfrak{F}^a)$.

Proof: If $R \in \mathfrak{F}^a$, then R is an S-transduction by definition, and the class of S-transductions is closed under union, concatenation, and concatenation closure $*$. Thus, $\mathfrak{R}(\mathfrak{F}^a)$ is contained in the class of S-transductions. On the other hand, any S-transduction is a member of $\mathfrak{R}(\mathfrak{F}^a)$ according to the preceding corollary. Hence, the claim follows.

4. Closure of the class of transductions under Pierce-product

It is straightforward to prove that the class of binary relations over Σ that are both LP and FAD is closed under the Pierce product (Proposition 4.11).

To extend this closure result to the larger class of all binary transductions over Σ, we first show that any transduction over Σ may be obtained from a LP FAD relation over the alphabet $\Sigma \cup \{\beta\}$, $\beta \notin \Sigma$, (Propositions 4.8, 4.4). With some added construction, proof of the closure for the larger class may then be reduced to that for the smaller.

Definition 4.1: A nondeterministic automaton $\mathfrak{A} = (S, \nu, s_I, D)$ is *elementary* iff $\nu \subseteq S \times \Sigma^n \times S$, i.e., the label of each edge is in Σ^n.

Definition 4.2: Given a relation $R \subseteq (\Sigma^*)^n$, we define the length-preserving relation $R_\beta \subseteq (\Sigma_\beta{}^*)^n$ by:

$$(u_1\beta^{m-l_{u_1}}, \cdots, u_n\beta^{m-l_{u_n}}) \in R_\beta \Leftrightarrow (u_1, \cdots u_n) \in R$$

$$\wedge\ m = \max_{1\le i\le n} \{l_{u_i}\} \quad \text{where} \quad \beta^n \quad \text{means} \quad \underbrace{\beta \cdots \beta}_{n}.$$

That is, R_β is obtained by taking n-tuples $(u_1, \cdots, u_n)$ of R and concatenating the least number of β's to the right hand ends of each u_i, $1 \le i \le n$ so as to make the resulting n-tuple $(v_1, \cdots, v_n)$ LP.

Definition 4.3: A relation $R \subseteq (\Sigma^*)^n$ is FAD iff there exists an *elementary* n-input NDA $\mathfrak{A}$ *over* Σ_β such that $T(\mathfrak{A}) = R_\beta$.

Making use of Theorem 11 of [RS], one may show that R is a relation defined by an elementary NDA iff there exists a multi-input finite state automaton that accepts R. Thus, the above definition of an FAD relation is equivalent to the one given in Section 2.

Proposition 4.4: Given an elementary (n-input) NDA $\mathfrak{A}$ over Σ, $T(\mathfrak{A})$ is LP and FAD. Conversely, if $R \subseteq (\Sigma^*)^n$ is FAD *and* length preserving, there exists an elementary NDA $\mathfrak{B}$ over Σ that defines R.

Proof: Given the NDA $\mathfrak{A}$ over Σ, consider it as defined over Σ_β. Since $T(\mathfrak{A})$ is length preserving, $T(\mathfrak{A}) = T(\mathfrak{A})_\beta$. Thus, $T(\mathfrak{A})$ is FAD.

If R is FAD, there exists an elementary NDA $\mathfrak{B}'$ over Σ_β such that $T(\mathfrak{B}') = R_\beta$. Since R is length preserving, $R_\beta = R$, and $\mathfrak{B}'$ defines R. Delete from $\mathfrak{B}'$ the edges with labels in which β occurs, to obtain elementary NDA $\mathfrak{B}$ over Σ. Then, $R = T(\mathfrak{B})$ as was desired to show. □

Definition 4.5: Consider the "augmented" alphabet

$$\Sigma_{\beta_1,\cdots,\beta_k} = \Sigma \cup \{\beta_1, \cdots, \beta_k\},$$

$$\beta_1 \ne \beta_2 \ne \beta_k, \qquad \beta_j \notin \Sigma, \qquad 1 \le j \le k.$$

The *deletion mapping* $d_{\beta_1,\cdots,\beta_k}$ is the homomorphism that carries $(\Sigma_{\beta_1,\cdots,\beta_k})^*$ into Σ^* determined by the requirement:

$$d_{\beta_1,\cdots,\beta_k}(\beta_j) = \Lambda, \qquad 1 \le j \le k$$

$$d_{\beta_1,\cdots,\beta_k}(\sigma) = \sigma, \qquad \sigma \in \Sigma.$$

Where ambiguity is no problem, the subscripts will be suppressed. Since d is a function, we observe that the relation $d^c \circ d$ over Σ_β^* is an equivalence relation and that

$$(u, v) \in d^c \circ d \Leftrightarrow d(u) = d(v) \in \Sigma^*.$$

Each equivalence class $[u]$ of $d^c \circ d$ contains a uniquely distinguished member $u \in \Sigma^*$.

Definition 4.6: The deletion mapping is extended to n-tuples $(u_1, \cdots, u_n) \in (\Sigma_\beta^*)^n$:

$$d(u_1, \cdots, u_n) = (d(u_1), \cdots, d(u_n)),$$

and to relations $R \subseteq (\Sigma_\beta^*)^n$:

$$d(R) = \{d(u_1, \cdots, u_n) : (u_1, \cdots, u_n) \in R\}.$$

Remark 4.7: It may be verified that for a binary relation R, $R \subseteq (\Sigma_\beta^*)^2$: $d(R) = d \circ R \circ d^c$ where the d (resp. d^c) on the right is a subset of $\Sigma_\beta^* \times \Sigma^*$ (resp. $\Sigma^* \times \Sigma_\beta^*$).

Proposition 4.8: Given the relation $R \subseteq (\Sigma^*)^n$, R is a transduction iff there exists an *elementary* nondeterministic n-input automaton $\mathfrak{A}$ over Σ_β such that:

$$R = d(T(\mathfrak{A})).$$

Proof $\Leftarrow$: Given $\mathfrak{A}$, obtain the NDA $\mathfrak{A}'$ by "replacing in the labels of $\mathfrak{A}$ all occurrences of β by Λ". Then,

$$T(\mathfrak{A}') = d(T(\mathfrak{A}))$$

which is a transduction.

Proof $\Rightarrow$: Given a transduction R, let $\mathfrak{A}$ be the NDA that defines R. Construct $\mathfrak{A}'$ from $\mathfrak{A}$ by replacing each edge labeled $(\sigma_{11} \cdots \sigma_{1m_1}, \cdots, \sigma_{n1} \cdots \sigma_{nm_n})$ by a sequence of edges respectively labeled:

$$(\sigma_{11}, \underbrace{\beta, \cdots \beta}_{n-1}) \cdots (\sigma_{1m_1}, \beta, \cdots, \beta)$$

$$(\beta, \sigma_{21}, \underbrace{\beta, \cdots \beta}_{n-2}) \cdots (\beta, \sigma_{2m_2}, \beta, \cdots \beta)$$

$$\vdots$$

$$(\underbrace{\beta, \cdots \beta}_{n-1}, \sigma_{n1}) \cdots (\beta, \cdots \beta, \sigma_{nm_n}).$$

Increment the collection of states appropriately, leaving, however, s_I, D unchanged. Then:

$$R = T(\mathfrak{A}) = d(T(\mathfrak{A}')),$$

where $\mathfrak{A}'$ is an elementary NDA over Σ_β. □

Corollary 4.9: If $R \subseteq (\Sigma^*)^n$ is FAD, then R is a transduction.

Proof: From the definition, there exists an elementary NDA $\mathfrak{A}$ over Σ_β such that $T(\mathfrak{A}) = R_\beta$. Then, $R = d(T(\mathfrak{A}))$ and R is a transduction. □

Corollary 4.10: If $R \subseteq (\Sigma^*)^2$ is a transduction and U is an FAD set, then the restriction $U \upharpoonright R$ is a transduction.

Proof: Let $\mathfrak{A}$ be the elementary NDA over Σ_β such that $R = d(T(\mathfrak{A}))$, then $T(\mathfrak{A})$ is FAD and length preserving. On the other hand, let $\mathfrak{B}'$ be the finite-state automaton that defines U.

Modify $\mathfrak{B}'$ as follows. Each edge (s, σ, s') of $\mathfrak{B}'$ is to be replaced by a collection of edges $\{(s, \sigma, \sigma', s') : \sigma' \in \Sigma_\beta\}$ and for all s, the set of edges $\{(s, \beta, \sigma', s) : \sigma' \in \Sigma_\beta\}$ added.

Denote the elementary NDA thus obtained by $\mathcal{B}$. $T(\mathcal{B})$ is FAD and LP. It may be verified that:

$$d(T(\mathcal{A}) \cap T(\mathcal{B})) = U \upharpoonright R.$$

But the intersection of LP FAD transductions is FAD and LP so that the relation $T(\mathcal{A}) \cap T(\mathcal{B})$ is defined by an elementary NDA over Σ_β (Proposition 4.4). It follows that $U \upharpoonright R$ is a transduction. □

Proposition 4.11: The class of (binary) FAD relations in $(\Sigma^*)^2$ is closed under Pierce product.

Proof: Follows as an immediate consequence of Theorem 10.

Definition 4.12: Define the partial ordering $\leq$ of Σ_β^* by:

$$u \leq v \Leftrightarrow \exists\ u_1, \cdots u_m \in \Sigma_\beta^*; n_0, \cdots, n_m \geq 0$$
$$[u = u_1 \cdots u_m \wedge v = \beta^{n_0} u_1 \beta^{n_1} u_2 \cdots u_m \beta^{n_m}]$$

i.e., $u \leq v$ iff one may obtain u by deleting *some* β's from v. Note that:

$$d(u) = d(v) \Rightarrow \exists\ w[u \leq w \wedge v \leq w].$$

Theorem 4.13: The class of binary transductions is closed under Pierce product.

Proof: Consider transductions $R_1, R_2 \subseteq (\Sigma^*)^2$. By Proposition 4.8, there exist elementary NDA's $\mathcal{A}_1'$, $\mathcal{A}_2'$ over Σ_β such that:

$$R_i = d(T(\mathcal{A}_i')) \qquad i = 1, 2.$$

Modify these automata as follows: to the collection of edges of $\mathcal{A}_i'$, $i = 1, 2$ add the set $\{(s, \beta, \beta, s) : s \in S_i\}$, i.e., "loops of unit lengths" labeled (β, β). Call the modified automata $\mathcal{A}_i$, and let $L_i \overset{\text{def}}{=} T(\mathcal{A}_i)$, then:

$$d(L_i) = R_i \qquad i = 1, 2,$$

and, further, L_i, $i = 1, 2$, satisfy

$$(u, v) \in L_i \wedge v \leq y \Rightarrow \exists x[u \leq x \wedge (x, y) \in L_i] \quad (1)$$

$$(u, v) \in L_i \wedge u \leq x \Rightarrow \exists y[v \leq y \wedge (x, y) \in L_i] \quad (2)$$

Recall $R_i = d \circ L_i \circ d^c$, $i = 1, 2$ and consider:

$$R_2 \circ R_1 = d \circ L_2 \circ d^c \circ d \circ L_1 \circ d^c$$
$$= d(L_2 \circ d^c \circ d \circ L_1).$$

We now show that $d(L_2 \circ d^c \circ d \circ L_1) = d(L_2 \circ L_1)$. Since $d^c \circ d$ contains the diagonal of $(\Sigma_\beta^*)^2$, $L_2 \circ L_1 \subseteq L_2 \circ d^c \circ d \circ L_1$, and, consequently, $d(L_2 \circ L_1) \subseteq d(L_2 \circ d^c \circ d \circ L_1)$. Take $(u, v) \in L_2 \circ d^c \circ d \circ L_1$. Then, there exist $s, t \in \Sigma_\beta^*$ such that:

$$(u, s) \in L_1$$
$$(s, t) \in d^c \circ d$$
$$(t, v) \in L_2.$$

But, $(s, t) \in d^c \circ d \Rightarrow d(s) = d(t) \Rightarrow \exists\ w\,[s \leq w \wedge t \leq w]$ and using properties (1), (2):

$$(u, s) \in L_1 \wedge s \leq w \Rightarrow \exists x[u \leq x \wedge (x, w) \in L_1]$$
$$(t, v) \in L_2 \wedge t \leq w \Rightarrow \exists y[v \leq y \wedge (w, y) \in L_2]$$

Thus:

$$(u, v) \in L_2 \circ d^c \circ d \circ L_1$$
$$\Rightarrow \exists x, y[u \leq x \wedge v \leq y \wedge (x, y) \in L_2 \circ L_1]$$
$$\Rightarrow \exists x, y[(x, y) \in L_2 \circ L_1 \wedge d(u, v) = d(x, y)]$$

and $d(L_2 \circ d^c \circ d \circ L_1) \subseteq d(L_2 \circ L_1)$. Hence, $d(L_2 \circ d^c \circ d \circ L_1) = d(L_2 \circ L_1)$. Finally, since $R_2 \circ R_1 = d(L_2 \circ d^c \circ d \circ L_1)$,

$$R_2 \circ R_1 = d(L_2 \circ L_1).$$

Now, L_i are LP FAD relations since they are defined by the elementary NDA's $\mathcal{A}_i$ over Σ_β, $i = 1, 2$ (Proposition 4.4). The composite $L_2 \circ L_1$ is then an FAD relation (Proposition 4.11) that is LP so that there exists (Proposition 4.4) an elementary NDA $\mathcal{B}$ over Σ_β that defines $L_2 \circ L_1$. Since $R_2 \circ R_1 = d(T(\mathcal{B}))$, we conclude $R_2 \circ R_1$ is a transduction (Proposition 4.8). □

Corollary 4.14: Let $\mathfrak{T}_\Sigma$ denote the class of all binary transductions over Σ.

Let $\mathfrak{C}_\Sigma$ denote the smallest class of binary relations over Σ closed under Pierce product and conversion which are that or, contains the LP transductions and the homomorphisms. Then:

$$\mathfrak{C}_{\Sigma_\beta} \supseteq \mathfrak{T}_\Sigma \supseteq \mathfrak{C}_\Sigma$$

Proof: Since homomorphism is a (1-state) transduction, and the class of binary transductions is closed under conversion and, by virtue of this theorem, Pierce product:

$$\mathfrak{T}_\Sigma \supseteq \mathfrak{C}_\Sigma$$

The other inclusion follows by Proposition 4.8 and Remark 4.7. □

Corollary 4.15: The following subclasses of binary transductions are closed under Pierce product: (a) locally finite transductions and symmetrically locally finite transductions; (b) S-transductions; (c) bounded transductions; (d) LP transductions; and (e) 1 : 1 functional transductions.

Proof: The defining properties of these classes (other than being collections of transductions) are preserved under the Pierce product. Since transductions are closed under the Pierce product, the corollary follows.

5. Decomposition of S-transductions

In this section we show that an S-transduction over Σ may be expressed as a Pierce product of simpler transductions over an augmented alphabet Σ'.

Theorem 5.1: The class of S-transductions over Σ is precisely the class of transductions $R = h \circ T$, where T is a LP *transduction and h is a homomorphism, and both relations T, h are over some $\Sigma' \supseteq \Sigma$. Further, if R is functional, then T may be chosen functional.*

Proof: Consider the first assertion of the theorem. Since LP transductions and homomorphisms are S-transductions and the class of S-transductions is closed under Pierce product, transductions $h \circ T$ are S-transductions.

Let R be an S-transduction. Denoting by $\mathfrak{F}^a$ the collection of those finite relations in $(\Sigma^*)^2$ which contain only admissible pairs, we have previously established (Corollary 3.20) that $R \in \mathfrak{R}(\mathfrak{F}^a)$. This implies that there exists an NDA $\mathcal{A}$ over Σ such that $R = T(\mathcal{A})$ and that all edges of $\mathcal{A}$ are labeled by admissible pairs.

Let $\{(u_1, v_1), \cdots, (u_n, v_n)\}$ be the set of labels that appear on the edges of $\mathcal{A}$. Construct an NDA $\mathcal{A}'$ over $\Sigma \cup \{0, 1, \cdots, n\}$ by changing the labels of the edges of $\mathcal{A}$ as follows. Edges labeled (u_i, v_i), $1 \leq i \leq n$ in $\mathcal{A}$ will be relabeled:

$$(u_i, i0^{lu_i - 1}) \quad \text{if} \quad (u_i, v_i) \neq \Lambda$$

$$(0, 0) \quad \text{if} \quad (u_i, v_i) = \Lambda$$

The procedure is proper since the labels of $\mathcal{A}$ are admissible pairs. Since the resulting labels of $\mathcal{A}'$ are all LP, $T(\mathcal{A}')$ is necessarily LP.

Define the homomorphism $h : \{0, 1, \cdots n\} \to \Sigma^*$ by the requirement:

$$h(0) = \Lambda$$

$$h(i) = v_i, \qquad 1 \leq i \leq n.$$

It is then immediate that $R = h \circ T(\mathcal{A}')$. Now let $\Sigma' = \Sigma \cup \{0, 1, \cdots, n\}$ and extend h so that it is included in $(\Sigma')^* \times (\Sigma')^*$. Taking $T \overset{\text{def}}{=} T(\mathcal{A}')$ completes proof of the first assertion.

To show the *second* assertion, we observe that since all edges of $\mathcal{A}'$ had LP labels, one may construct, by "appropriately subdividing edges of $\mathcal{A}'$ ", an *elementary* NDA $\mathcal{A}''$ such that $T = T(\mathcal{A}') = T(\mathcal{A}'')$. Then, T is FAD (Proposition 4.4).

Now let R be functional. T may not be functional, but it is FAD and we invoke the following result (Lemma 6.5, [CCE]):

If T is a binary relation that is FAD, then there exists an FAD *function* T' such that $T' \subseteq T$ and dom T' = dom T.

Consider $h \circ T'$. Since $T' \subseteq T$, $h \circ T' \subseteq R$. On the other hand, $R \subseteq h \circ T'$. To see this, take any $(u, v) \in R$. Then there exists w such that $(u, w) \in T$ and, since dom T' = dom T, there exists w' such that $(u, w') \in T'$. Then, $(u, h(w')) \in h \circ T' \subseteq R$. But R is functional, hence $h(w') = v$ and $(u, v) \in h \circ T'$. In the resulting decomposition $R = h \circ T'$, T' is a transduction (Corollary 4.9) and since $T' \subseteq T$, T' is LP. □

Corollary 5.2: For all homomorphisms h_0, LP transductions L_0 over Σ there exist homomorphism h, LP transduction L over $\Sigma' \supseteq \Sigma$ such that:

$$L_0 \circ h_0 = h \circ L.$$

Proof: L_0, h_0 are S-transductions over Σ and so is $L_0 \circ h_0$. From the theorem the corollary follows.

Remark 5.3: The converse claim to the corollary above is not true. That is, there exist homomorphism h, LP transduction L such that for all homomorphisms h_0, LP transductions L_0:

$$h \circ L \neq L_0 \circ h_0.$$

The reason for this is that all sets $\{v : (u, v) \in L_0 \circ h_0, u \in \Sigma^*\}$ have the property that they do not contain sequences of unequal lengths; whereas, in general, a set $\{v : (u, v) \in h \circ L, u \in \Sigma^*\}$ may contain sequences of different lengths.

Proposition 5.4: Given an FAD relation $R \subseteq (\Sigma^*)^2$, $D \subseteq$ dom R:

(a) the maximal subdomain D such that $D \upharpoonright R$ is LP is FAD;
(b) the maximal subdomain D such that $D \upharpoonright R$ is functional is FAD;
(c) the maximal subdomain D such that $D \upharpoonright R$ is the identity mapping is FAD.

Proof: These contentions follow from Theorem 10.

Remark 5.5: The preceding proposition does not hold for the class of S-transductions.

Consider homomorphisms $h_1 = \{(0, 00), (1, \Lambda)\}^*$, $h_2 = \{(0, \Lambda), (1,00)\}^*$. For the S-transduction $h_1 \cup h_2$, the maximal subdomains D of (a), (b) in the above proposition are equal to dom $(h_1 \cap h_2)$. This set is the collection of those sequences u for which the number of occurrences of 0's and 1's in u are equal, hence not FAD.

Similarly, consider the S-transduction $R = \{(0, 00)\}^* \{(0, 1)\}^* \{(1, \Lambda)\}^*$. The maximal subdomain on which R is the identity is the set $\{0^{2n}1^n : n \geq 0\}$ which is not FAD.

6. Subclasses of transductions; bounded transductions

The diagram of Figure 1 shows how the various subclasses of binary transductions are related under (set theoretic) inclusion. In this diagram the small unlabeled circles are meant to emphasize that we have not named certain unions and intersections of classes. All the diagrammed relations can be established either by definition or from simple counterexamples, except for the claim that FAD relations are transductions (Corollary 4.9) and the claim that is the concern of this section, given in the following theorem.

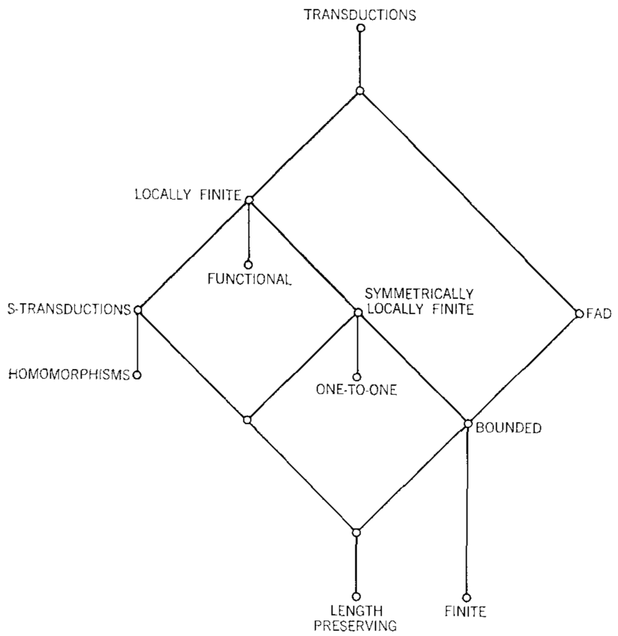

Figure 1 Subclasses of transductions

Theorem 6.1: The intersection of the class of "symmetrically locally finite" transductions and the class of FAD *relations is exactly the class of bounded transductions.*

Proof (Part 1): For the *first part* of the proof, we use the following lemma.

Lemma 6.2: Given a relation $R \subseteq (\Sigma^*)^2$: R is FAD and locally finite $\Rightarrow \exists\ m \forall (u, v)\ [(u, v) \in R \Rightarrow l_v - l_u < m]$.

Proof: Let $\mathcal{A}$ be the elementary NDA over Σ_β that defines R_β, and let n be the number of states of $\mathcal{A}$. Assume, to the contrary, that $\forall m\ \exists\ (u, v) \in R[l_v - l_u \geq m]$ and, in particular, $\exists\ (u, v) \in R[l_v - l_u > n]$; that is, $\exists\ (u\beta^p, v) \in R_\beta : p > n$. Write $(u\beta^p, v) = (u, x)\ (\beta^p, y)$ where both factors† are LP. Since the length of (β^p, y) exceeds the number of states of $\mathcal{A}$, if a path labeled (β^p, y) connects s_0, s_p and passes through $s_1, \cdots s_{p-1}$, then there will exist some state repeated in the sequence $s_0, s_1, \cdots s_p$.

From this remark we may immediately conclude the following: If a path in $\mathcal{A}$ labeled $(u, x)\ (\beta^p, y)$ connects states s, s', then for an arbitrary integer q there will exist $r > q$, $y' \in \Sigma^*$ such that a path in $\mathcal{A}$ labeled $(u, x)\ (\beta^r, y')$ connects s, s'. Thus, $\mathcal{A}$ defines an infinity of pairs of the form $(u, x)\ (\beta^r, y')$ and $R = d(T(\mathcal{A}))$ is not locally finite, producing a contradiction. This proves the lemma.

The *first part* of the proof of the theorem may now be completed as follows. Since R is FAD, so is R^c and using the lemma:

R is FAD and locally finite $\Rightarrow \exists\ m_1\ \forall (u, v)\ [(u, v) \in R \Rightarrow l_v - l_u < m_1]$;

R^c is FAD and locally finite $\Rightarrow \exists\ m_2\ \forall (u, v)\ [(v, u) \in R \Rightarrow l_u - l_v < m_2]$.

Finally, $(u, v) \in R \Rightarrow l_v - l_u < \max(m_1, m_2) \wedge l_u - l_v < \max(m_1, m_2)$ so that R is bounded.

Proof (Part 2): In general, if R is bounded, both R, R^c are locally finite. We assume R is a bounded *transduction* and show R is FAD. Since R is a transduction, $R \in \mathfrak{K}(\mathfrak{F})$ (Proposition 3.5). Proof is by induction, and we use the following observation. Given nonempty relations S, $T \subseteq (\Sigma^*)^2$:

S is bounded $\wedge\ T$ is bounded $\Leftrightarrow S \cup T$ is bounded $\Leftrightarrow ST$ is bounded.

For $R \in \mathfrak{F}_0$, R is a finite relation and necessarily FAD. Let S, $T \in \mathfrak{F}_i$ and consider the cases:

(a) $R = S \cup T$. Since $S \cup T$ is bounded, both S, T are bounded. By the inductive assumption, both S, T are FAD. Hence, R is FAD.

(b) $R = S^*$. Since S^* is bounded, S is LP and, by the inductive assumption, S is FAD. That $R = S^*$ is FAD (and, incidentally, LP) is then a consequence of the following lemma.

Lemma 6.3: The class of n-ary relations over Σ that are both FAD and LP is precisely $K(\mathfrak{F}^{\text{LP}})$ where $\mathfrak{F}^{\text{LP}}$ is the class of finite, LP relations.

Proof: It follows from Definition 4.3 of an FAD relation (see discussion following that definition) that this lemma is essentially equivalent to Theorem 14 of [RS].

(c) $R = ST$. Since ST is bounded, both S, T are bounded (assuming $ST \neq \phi$). By the inductive assumption, both S, T are FAD. That $R = ST$ is FAD is a consequence of the following lemma.

Lemma 6.4: Given relations S, $T \subseteq (\Sigma^*)^2$. If S is FAD and *bounded* and T is FAD, then ST is FAD.

Proof: Since S, T are FAD, so are the LP relations S_β, T_β, and $S_\beta T_\beta$. From Theorem 10,† it follows that if a relation $R \subseteq (\Sigma_\beta^*)^n$ is FAD, then the relation R', obtained by truncating in each n-tuple of R the final string of β's from all components of the n-tuple, is FAD. In particular, $S_\beta T$ is FAD. It is then sufficient to show, *due to the boundedness of S*, that $(S_\beta T)'$ is FAD, where $(S_\beta, T)'$ is obtained by deleting in the first components of all pairs of $S_\beta T$ the leftmost occurrence of a β (if no such β occurs, the pair is taken unchanged).

† Given $u, v \in (\Sigma^*)^n$, we say v is a factor of u iff there exist $x, y \in (\Sigma^*)^n$ such that $u = x \cdot v \cdot y$.

† See Appendix, Section 10. In this use of the theorem we let Σ_β play the role of Σ and introduce a new letter to play that β.

By virtue of Theorem 10, there exists a formula F of L such that $(u, v) \in S_\beta T \Leftrightarrow F[\mathbf{u}, \mathbf{v}]$. Consider the following formula $F'[\mathbf{u}', \mathbf{v}]$ of L:

$$F'[\mathbf{u}', \mathbf{v}] = \exists \mathbf{u}[F[\mathbf{u}, \mathbf{v}] \wedge ([1] \vee [2])]$$

where:

$$[1] = \exists t[\mathbf{u}(t) = \beta \wedge \forall t'[t' < t \Rightarrow \mathbf{u}(t) \neq \beta]$$
$$\wedge \forall t'[t' < t \Rightarrow \mathbf{u}'(t') = \mathbf{u}(t')]$$
$$\wedge \forall t'[t' \geq t \Rightarrow \mathbf{u}(t' + 1) = \mathbf{u}'(t')]]$$
$$[2] = \forall t[\mathbf{u}(t) \neq \beta \wedge \mathbf{u}'(t) = \mathbf{u}(t)]$$

Clearly, $(u', v) \in (S_\beta T)' \Leftrightarrow F'[\mathbf{u}', \mathbf{v}]$ and, thus, $(S_\beta T)'$ is FAD. This completes proof of the lemma.

We may conclude that if the theorem holds for all $R \in \mathfrak{F}_i$, it holds for all $R \in \mathfrak{F}_{i+1}$, which completes the induction. □

Corollary 6.5: Given a relation $R \subseteq (\Sigma^*)^2$. If R^* is FAD and symmetrically locally finite, then R^* (and R) is LP.

Proof: Under the assumptions of the corollary, we may conclude from the theorem that R^* is bounded, and the conclusion follows.

Corollary 6.6: Given a transduction R. If R is LP, then R is FAD.

Proof: Since R is LP, it is certainly bounded, and the conclusion follows by the theorem.

The preceding theorem holds for n-ary relations under the following definitions.

Definition 6.7: Given $R \subseteq (\Sigma^*)^n$ define the relations $R_1^{(u)}, \cdots R_n^{(u)}$ for all $u \in \Sigma^*$ by:

$$(u_1, \cdots u_n) \in R_i^{(u)} \Leftrightarrow (u_1, \cdots u_n) \in R \wedge u_i = u \qquad 1 \leq i \leq n.$$

A relation $R \subseteq (\Sigma^*)^n$ is *symmetrically locally finite* iff $\forall u \forall_1^n i[R_i^{(u)}$ is finite].

A relation $R \subseteq (\Sigma^*)^n$ is *bounded* iff there exists an m such that:

$$(u_1, \cdots u_n) \in R \Rightarrow \forall_1^n i, j[l_{u_i} - l_{u_j} < m].$$

Proposition 6.8: Given a homomorphism h and the subset $X \subseteq \Sigma^*$, let pref (X) denote the smallest set that contains X and is prefix-closed. Then: X is FAD $\wedge X\upharpoonright h$ is bounded $\Rightarrow$ pref $(X)\upharpoonright h$ is bounded.

Proof: We use the following observations:

1) Given sets $Y, Z \subseteq \Sigma^*$:

$$\text{pref } (Y \cup Z) = \text{pref } (Y) \cup \text{pref } (Z)$$
$$\text{pref } (Y^*) = Y^* \text{ pref } (Y)$$
$$\text{pref } (YZ) = \text{pref } (Y) \cup Y \text{ pref } (Z)$$

2) Given a relation $R \subseteq (\Sigma^*)^2$ and sets $Y, Z \subseteq \Sigma^*$:

$$(Y \cup Z) \upharpoonright R = Y \upharpoonright R \cup Z \upharpoonright R;$$

if, further, R is a *homomorphism* of F_Σ:

$$(YZ) \upharpoonright R = (Y \upharpoonright R)(Z \upharpoonright R).$$

Since X is FAD, $X \in \mathfrak{R}(\mathfrak{F})$ and the proof proceeds by induction. If $X \in \mathfrak{F}$, pref X is finite and pref $X\upharpoonright h$ necessarily bounded.

Let $Y, Z \in \mathfrak{F}_i$ and consider the cases:

(a) $X = Y \cup Z$. Since $X\upharpoonright h = Y\upharpoonright h \cup Z\upharpoonright h$ is bounded, so are $Y\upharpoonright h$, $Z\upharpoonright h$. By the inductive assumption, pref $Y\upharpoonright h$, pref $Z\upharpoonright h$ are bounded. Since: pref $Y\upharpoonright h \cup$ pref $Z\upharpoonright h =$ (pref $Y \cup$ pref $Z)\upharpoonright h =$ pref $(Y \cup Z)\upharpoonright h$, pref $X\upharpoonright h$ is bounded.

(b) $X = YZ$. Since h is a homomorphism, $X\upharpoonright h = (Y\upharpoonright h)(Z\upharpoonright h)$, and since $X\upharpoonright h$ is bounded, so are $Y\upharpoonright h$, $Z\upharpoonright h$. By the inductive assumption pref $Y\upharpoonright h$, pref $Z\upharpoonright h$ are bounded. Using the fact that h is a homomorphism:
pref $Y\upharpoonright h \cup (Y\upharpoonright h)($pref $Z\upharpoonright h) =$ pref $Y\upharpoonright h \cup Y$ pref $Z\upharpoonright h$ (pref $Y \cup Y$ pref $Z)\upharpoonright h =$ pref $(YZ)\upharpoonright h$.
Hence, pref $X\upharpoonright h$ is bounded.

(c) $X = Y^*$. Since $Y^*\upharpoonright h$ is assumed bounded, and $Y \subseteq Y^*$, $Y\upharpoonright h$ is bounded and, by the inductive assumption, pref $Y\upharpoonright h$ is bounded. Since h is a homomorphism $(Y^*\upharpoonright h)($pref $Y\upharpoonright h) = (Y^*$ pref $Y)\upharpoonright h =$ pref $(Y^*)\upharpoonright h$; hence, pref $X\upharpoonright h$ is bounded.

Thus, if the proposition holds for $\mathfrak{F}_i$, it holds for $\mathfrak{F}_{i+1}$, which completes the induction. □

Given $X \subseteq \Sigma^*$, define suff (X) as the smallest set Y such that $\rho(Y)$ is prefix-closed and $X \subseteq Y$. Observe:

(a) suff $= \rho \circ$ pref $\circ\, \rho$
(b) suff $\circ$ pref $=$ pref $\circ$ suff
(c) given $X \subseteq \Sigma^*$, suff $\circ$ pref (X) is the set of all factors of all members of X.

The preceding proposition holds when pref (X) is replaced by suff (X), and one concludes:

Proposition 6.9: Given a homomorphism h of F_Σ and $X \subseteq \Sigma^*$, let Y be the set of all factors of members of X. Then: X is FAD $\wedge X\upharpoonright h$ is bounded $\Rightarrow Y\upharpoonright h$ is bounded.

Proposition 6.10: If a homomorphism h is FAD, then either h is LP or *ran* $h = \Lambda$.

Proof: Observe that for any $\sigma \in \Sigma$, $h(\sigma) \in \Sigma$ or $h(\sigma) = \Lambda$. For, assume to the contrary that there exists some σ_0 such that $h(\sigma_0) = u \wedge l(u) \geq 2$. Then, consider the restriction $\{\sigma_0\}^*\upharpoonright h = \{(\sigma_0, u)\}^*$ which is both FAD and symmetrically locally finite. By Corollary 6.5, (σ_0, u) must be LP, and $l(u) = 1$ producing a contradiction.

Next, assume that there exist $\sigma_0, \sigma_1, \in \Sigma$ such that

$h(\sigma_0) = \sigma \in \Sigma$ and $h(\sigma_1) = \Lambda$. Then, $\{(\sigma_0\sigma_1)\}^* \upharpoonright h = \{(\sigma_0\sigma_1, \sigma)\}^*$ is again both FAD and symmetrically locally finite, and as before, Corollary 6.5 yields a contradiction. Thus, there remain the cases:

1) For all σ, $h(\sigma) \in \Sigma$ and h is LP;

2) For all σ, $h(\sigma) = \Lambda$ and $ran\ h = \Lambda$.

7. Decomposition of LP transductions

Thus far we know that the class of LP transductions is precisely the class of LP FAD relations (6.6) and that it is closed under all the operations on relations given in Table 1. (See 4.11, 6.3, and 6.6.) In this section we show that in a manner analogous to the decomposition of S-transductions, LP transductions over Σ may be expressed as the Pierce product of simpler transductions over an augmented alphabet Σ'.

We first characterize sequential relations and then show that every LP FAD relation (resp. function) is a composite of a sequential relation (resp. function) and the reversal of a sequential relation (resp. function). The argument for the parenthetical statement (Theorem 7.8) is substantially more involved than that for the other two statements (Propositions 7.1, 7.4).

Proposition 7.1: Let $\mu \subseteq \Sigma^* \times \Sigma^*$ be a LP relation.

1) If μ is a sequential relation, then μ is nonempty, prefix-closed and FAD.
2) Conversely, if μ is nonempty, prefix-closed and FAD, then μ is a sequential relation.
3) If, in addition, μ is functional, then μ is the associated function of a sequential machine.

Proof:

1) If μ is a sequential relation, then it is the associated relation of some NDSM $\mathfrak{M} = (S, \nu, s_I)$, $\nu \subseteq (S \times \Sigma) \times (\Sigma \times S)$. It is immediate from the definition of NDSM that μ is nonempty and prefix-closed.

Consider the elementary NDA $\mathfrak{A} = (S, \nu, s_I, D)$, $D = S$ where ν of $\mathfrak{A}$ is ν of $\mathfrak{M}$ regarded as a subset of $S \times (\Sigma \times \Sigma) \times S$. [We identify elements of Σ with the elements of Σ^* of unit length. Thus, $S \times (\Sigma \times \Sigma) \times S \subseteq S \times (\Sigma^* \times \Sigma^*) \times S$.] Then, $\mu = T(\mathfrak{A})$ and thus is FAD (4.4 or 6.6).

2) Suppose $\mu \neq \phi$, FAD and prefix-closed. There exists an elementary NDA $\mathfrak{A} = (S, \nu, s_I, D)$ such that $\mu = T(\mathfrak{A})$, (4.4). Further, ν may be chosen functional with domain $S \times (\Sigma \times \Sigma)$. Let $\mathfrak{A}' = (D, \nu', s_I, D)$ where $\nu' = \nu \cap (D \times (\Sigma \times \Sigma) \times D)$. We claim $T(\mathfrak{A}') = T(\mathfrak{A})$. Note, that since $\mu \neq \phi$ and prefix-closed, $\Lambda \in \mu$ and $s_I \in D$. Suppose $u \in T(\mathfrak{A})$, then there is a successful path p in $\mathfrak{A}$ with label u. Now, for any LP $u' \in \Sigma^* \times \Sigma^*$ there exists a *unique* path p' in $\mathfrak{A}$, labeled u' and beginning in s_I. Thus, if u' is a prefix of u, p' is a prefix of p. Since μ is prefix-closed, $u' \in T(\mathfrak{A})$ and p' terminates in a state $s \in D$. Thus, every state through which p passes in $\mathfrak{A}$ is a state in D, hence p is also a path in $\mathfrak{A}'$ and $\mu \in T(\mathfrak{A}')$, which justifies the claim.

Reversing the procedure of (1), the desired NDSM is $\mathfrak{M} = (D, \nu', s_I)$ with ν' treated as a subset of $(D \times \Sigma) \times (\Sigma \times D)$. It is immediate that the sequential relation associated with $\mathfrak{M}$ is $T(\mathfrak{A}') = T(\mathfrak{A}) = \mu$, which concludes the proof.

3) It may be assumed about $\mathfrak{A}$, $\mathfrak{A}'$ of (2) that every state is accessible from s_I. Then, if μ is functional, ν' regarded as a subset of $(D \times \Sigma) \times (\Sigma \times D)$ is functional. For suppose $((s, \sigma), (\sigma_1, s_1)) \in \nu'$ and $((s, \sigma), (\sigma_2, s_2)) \in \nu'$. If $u \in \Sigma^* \times \Sigma^*$ is the label of a path (which exists by the accessibility assumption) that begins with s_I and terminates with s, then $u \in \mu$, $u \cdot (\sigma, \sigma_1) \in \mu$ and $u \cdot (\sigma, \sigma_2) \in \mu$. Since μ is functional, $\sigma_1 = \sigma_2$. Since $\nu' \subseteq (D \times (\Sigma \times \Sigma)) \times D$ is functional, $s_1 = s_2$. Hence, $\nu' \subseteq (D \times \Sigma) \times (\Sigma \times D)$ is functional. □

Definition 7.2: For purposes of this section, it is convenient to extend the notion of NDA to allow for several, rather than just one, initial states. Thus, an NDA will be the ordered quadruple $\mathfrak{A} = (S, \nu, D_I, D_F)$ where $D_I, D_F \subseteq S$. A successful path will be one that begins with an element of D_I and ends with an element of D_F. *It may readily be verified that the class of transductions is not thereby increased.* When the notion NDSM is similarly changed to $\mathfrak{M} = (S, \nu, D_I)$, the class of sequential relations is not increased.

Definition 7.3: Given an NDA $\mathfrak{A} = (S, \nu, D_I, D_F)$, the *dual* $\mathfrak{A}^D$ of $\mathfrak{A}$ is the NDA $\mathfrak{A}^D = (S, \nu^D, D_F, D_I)$ where:

$$(s, u, s') \in \nu^D \Leftrightarrow (s', \rho(u), s) \in \nu$$

for $u \in (\Sigma^*)^n$. Then, the reversal of a (successful) path in $\mathfrak{A}$ is a (successful) path in $\mathfrak{A}^D$ and vice versa. See Theorem 12 in [RS].

Proposition 7.4: Let $R \subseteq \Sigma^* \times \Sigma^*$ be an LP relation. Then, R is FAD iff there exist sequential relations μ_1, μ_2 over an augmented alphabet $\Sigma' \supseteq \Sigma$ such that $R = \rho \circ \mu_2 \circ \rho \circ \mu_1$ $(= \rho(\mu_2) \circ \mu_1)$.

Proof: If μ_1, μ_2 are sequential relations, they are FAD (7.1) and $\rho(\mu_2)$ is FAD, as well as the composite $\rho(\mu_2) \circ \mu_1$ (4.11).

Assume the LP relation R is FAD so that there exists the elementary NDA $\mathfrak{A} = (S, \nu, s_I, D)$ such that $R = T(\mathfrak{A})$ (4.4). Derive the NDSM's $\mathfrak{M}_1' = (S, \nu, s_I)$, $\mathfrak{M}_2' = (S, \nu^D, D)$ associated with sequential relations μ_1', μ_2' respectively (with ν treated as a subset of $(S \times \Sigma) \times (\Sigma \times S)$). Clearly, $R = \rho(\mu_2') \cap \mu_1'$. We now modify $\mathfrak{M}_1', \mathfrak{M}_2'$ so as to obtain NDSM's $\mathfrak{M}_1, \mathfrak{M}_2$ with the associated relations μ_1, μ_2 such that:

$$(u, (u, v)) \in \mu_1 \Leftrightarrow (u, v) \in \mu_1',$$
$$((u, v), v) \in \mu_2 \Leftrightarrow (u, v) \in \mu_2';$$

that is,

$$((u, v), v) \in \rho(\mu_2) \Leftrightarrow (u, v) \in \rho(\mu_2').$$

Then,

$$(u, v) \in \rho(\mu_2) \circ \mu_1$$

$$\Leftrightarrow \exists\, w[(u, w) \in \mu_1 \wedge (w, v) \in \rho(\mu_2)]$$

$$\Leftrightarrow (u, (u, v)) \in \mu_1 \wedge ((u, v), v) \in \rho(\mu_2)$$

$$\Leftrightarrow (u, v) \in \mu_1' \wedge (u, v) \in \rho(\mu_2')$$

$$\Leftrightarrow (u, v) \in R \quad \square$$

Note that the relation μ_2 is functional, thus a sequential mapping (7.1).

In the summarizing section, after NDSM's were defined, they were restricted to be elementary. If this restriction is removed, i.e., ν permitted to be a subset of $(S \times \Sigma^*) \times (\Sigma^* \times S)$, the preceding argument can be employed to establish the following proposition.

Proposition 7.5: Let $R \subseteq \Sigma^* \times \Sigma^*$ be a (binary) transduction. There exist sequential relations μ_1, μ_2 (associated, in general, with nonelementary NDSM's) over an augmented alphabet Σ' such that:

$$R = \rho(\mu_2) \circ \mu_1.$$

In preparation for Theorem 7.8, we make the following definitions.

Definition 7.6: Given NDA $\mathcal{A} = (S, \nu, s_I, D)$, $P_{\mathcal{A}}$ is the converse to the relation obtained by restricting λ (Footnote to Proposition 3.5) to the set of successful paths in $\mathcal{A}$:

$$P_{\mathcal{A}} \overset{\text{def}}{=} \{(u, p) : u = \lambda(p) \wedge p \text{ is a successful path in } \mathcal{A}\}.$$

Definition 7.7: Given an elementary NDA $\mathcal{A} = (S, \nu, D_I, D_F)$ where $\nu \subseteq S \times \Sigma \times S$, the associated elementary NDA $\mathcal{A}^\tau$ is defined as $\mathcal{A}^\tau = (\mathfrak{P}(S), \nu^\tau, D_I, \mathfrak{P}(S))$ where $\mathfrak{P}(S)$ is the class of all subsets of S and ν^τ is defined by:

$$(S_1, \sigma, S_2) \in \nu^\tau \Leftrightarrow S_2 = \{s' : \exists s \in S_1[(s, \sigma, s') \in \nu]\}$$

for $S_1, S_2 \subseteq S$.

We remark that ν^τ is a function with domain $\mathfrak{P}(S) \times \Sigma$ and that $T(\mathcal{A}^\tau)$ is prefix-closed. In addition, we will employ the following property of $\mathcal{A}^\tau$. If $(S_1, \sigma_1, S_2) \cdots (S_m, \sigma_m, S_{m+1})$ is a path in $\mathcal{A}^\tau$ and s_{m+1} any member of S_{m+1}, then there exist $s_i \in S_i$, $1 \leq i \leq m$ such that $(s_1, \sigma_1, s_2) \cdots (s_m, \sigma_m, s_{m+1})$ is a path in $\mathcal{A}$.

Theorem 7.8: Given $F \subseteq \Sigma^* \times \Sigma^*$, FAD, LP, and functional. There exist sequential mappings (functions) F_1, F_2 over an augmented alphabet $\Sigma' \supseteq \Sigma$ such that:

$$F = \rho \circ F_2 \circ \rho \circ F_1 (= \rho(F_2) \circ F_1).$$

Proof:

1) Since F is FAD and LP, there exists an elementary NDA $\mathcal{A}' = (S, \nu', s_I, D)$ such that $F - T(\mathcal{A}')$, and $\nu' \subseteq (S \times \Sigma^2) \times S$ is functional.

Without loss of generality, assume every edge in ν' is part of a successful path in $\mathcal{A}'$, i.e., given any $(s, \sigma, \sigma', s') \in \nu'$ there exists a successful path $(s_1, \sigma_1, \sigma_1', s_2) \cdots (s_m, \sigma_m, \sigma_m', s_{m+1})$ and an i, $1 \leq i \leq m$ such that $(s_i, \sigma_i, \sigma_i', s_{i+1}) = (s, \sigma, \sigma', s')$.

Let the elementary NDA $\mathcal{A} = (S, \nu, s_I, D)$ be the elementary NDA derived from $\mathcal{A}'$ by deleting output symbols from the edges, i.e., such that $\nu \subseteq S \times \Sigma \times S$ is defined by:

$$(s, \sigma, s') \in \nu \Leftrightarrow \exists \sigma'[(s, \sigma, \sigma', s') \in \nu'].$$

It is a consequence of assuming that every edge in ν' is part of a successful path in $\mathcal{A}'$ and of the fact that F is *functional* that:

$$(s, \sigma, \sigma_1, s') \in \nu' \wedge (s, \sigma, \sigma_2, s') \in \nu' \Rightarrow \sigma_1 = \sigma_2$$

and that, therefore, two *functions* ψ, θ may be defined, $\psi : \nu \to \nu'$, $\theta : \nu \to \Sigma$ as follows:

$$\psi(s, \sigma, s') = (s, \sigma, \sigma', s') \Leftrightarrow (s, \sigma, \sigma', s') \in \nu'$$

$$\theta(s, \sigma, s') = \sigma' \Leftrightarrow (s, \sigma, \sigma', s') \in \nu'.$$

We extend θ to a LP homomorphism carrying ν^* into Σ^* to obtain in a straightforward manner:

$$F = \theta \circ P_{\mathcal{A}}.$$

Now ψ is a one-to-one correspondence between edges in ν, ν' respectively, and if ψ is extended to ν^*, it yields, in particular, a one-to-one correspondence between successful paths in $\mathcal{A}$, $\mathcal{A}'$ respectively. With the aid of ψ, we show that although $\nu \subseteq (S \times \Sigma) \times S$ is not in general functional, $P_{\mathcal{A}}$ is a function.

Consider successful paths p_1, p_2 in $\mathcal{A}$ with common label u. Then, $\psi(p_1)$, $\psi(p_2)$ are successful paths in $\mathcal{A}'$ labeled (u, v_1), (u, v_2) respectively. Since (u, v_1), (u, v_2) are then in F and F is functional, $v_1 = v_2 = v$. On the other hand, $\mathcal{A}'$ is elementary and $\nu' \subseteq (S \times \Sigma^2) \times S$ is functional so that the successful path in $\mathcal{A}'$ labeled (u, v) is unique: $\psi(p_1) = \psi(p_2)$. Since ψ is one-to-one, $p_1 = p_2$, and we have that a successful path in $\mathcal{A}$ labeled u is unique: $P_{\mathcal{A}}$ is functional. Clearly, $P_{\mathcal{A}}$ is a transduction for any NDA $\mathcal{A}$ and for an elementary NDA $\mathcal{A}$ it is LP so that $P_{\mathcal{A}}$ is FAD (6.6).

It now suffices to verify the theorem for the LP FAD function $P_{\mathcal{A}}$. Indeed, if $P_{\mathcal{A}} = \rho(P_2) \circ P_1$ where P_1, P_2 are sequential mappings:

$$F = \theta \circ \rho(P_2) \circ P_1 = \rho(\theta \circ P_2) \circ P_1$$

since θ is a homomorphism. Since θ is LP as well, if P_2 is a sequential mapping, then so is $\theta \circ P_2$. Then, taking $F_1 \overset{\text{def}}{=} P_1$, $F_2 \overset{\text{def}}{=} \theta \circ P_2$, the theorem follows.

2) Consider the elementary NDA $\mathcal{A}^\tau$ as defined in 7.7. Given a path $(S_1, \sigma_1, S_2) \cdots (S_n, \sigma_n, S_{n+1})$ in $\mathcal{A}^\tau$, $S_1 = \{s_I\}$ and any element $s \in S_{n+1}$, there exists a path in $\mathcal{A}$, beginning in s_I and terminating with s labeled $\sigma_1 \cdots \sigma_n$. In

particular, if $s \in S_{n+1} \cap D$, the path in $\mathcal{A}$ is *successful.*

3) Let $\mathcal{A}^{D\tau} \overset{\text{def}}{=} (\mathcal{A}^D)^\tau$ be the elementary NDA obtained by taking the dual automaton of $\mathcal{A}$ (7.3) and finding its associated NDA as in 7.7. If $(S_1, \sigma_1, S_2) \cdots (S_n, \sigma_n, S_{n+1})$ is the *reversal* of a path in $\mathcal{A}^{D\tau}$, $S_{n+1} = D$, then for any element $s \in S_1$ there exists a path in $\mathcal{A}$ beginning in s and terminating in an element of D labeled $\sigma_1 \cdots \sigma_n$. In particular, if $s = s_I$, the path in $\mathcal{A}$ is successful.

4) Consider $P_{\mathcal{A}}(u) = (s_1, \sigma_1, s_2) \cdots (s_n, \sigma_n, s_{n+1})$, the *unique* successful path in $\mathcal{A}$, $s_1 = s_I$, $s_{n+1} \in D$ labeled $u = \sigma_1 \cdots \sigma_n$. If $P_{\mathcal{A}^\tau}(u) = (S_1, \sigma_1, S_2) \cdots (S_n, \sigma_n, S_{n+1})$ is the (unique successful) path determined by u in $\mathcal{A}^\tau$, then $S_1 = \{s_I\}$ and $s_i \in S_i$, $1 \le i \le n+1$. If $\rho \circ P_{\mathcal{A}^{D\tau}} \circ \rho(u) = (S_1', \sigma_1, S_2') \cdots (S_n', \sigma_n, S_{n+1}')$ is the *reversal* of the (unique successful) path determined by $\rho(u)$ in $\mathcal{A}^{D\tau}$, then $S_{n+1}' = D$ and $s_i \in S_i'$, $1 \le i \le n+1$. In conclusion, $s_i \in S_i \cap S_i'$, $1 \le i \le n+1$. Moreover, these intersections contain a unique state, i.e., $\forall s_i'[s_i' \in S_i \cap S_i' \Rightarrow s_i' = s_i]$, $1 \le i \le n+1$. To show this, suppose, to the contrary, that $s_i' \in S_i \cap S_i' \wedge s_i' \ne s_i$. Then, $i > 1$ and, by (2), there is a path in $\mathcal{A}$ from s_I to s_i' with label $\sigma_1 \cdots \sigma_{i-1}$. By (3), there is a path in $\mathcal{A}$ from s_i' to an element of D labeled $\sigma_i \cdots \sigma_n$ (if $i = n+1$, then $s_i' \in D$ and this path is null). Concatenating the two paths produces a path in $\mathcal{A}$ labeled $u = \sigma_1 \cdots \sigma_n$ *distinct* from the originally given $P_{\mathcal{A}}(u)$, which is a contradiction.

We have, then, given $u \in \operatorname{dom} P_{\mathcal{A}}$, a means of recovering $P_{\mathcal{A}}(u)$ from $P_{\mathcal{A}^\tau}(u)$, $\rho \circ P_{\mathcal{A}^{D\tau}} \circ \rho(u)$ by "edgewise intersections".† Next, we obtain the desired sequential functions using $\mathcal{A}^\tau$, $\mathcal{A}^{D\tau}$ respectively.

5) Let $\mathcal{A}^{D\tau}$ be $\mathcal{A}^{D\tau} = (\mathfrak{P}(S), \nu^{D\tau}, D, \mathfrak{P}(S))$ on Σ, where $\nu^{D\tau} \overset{\text{def}}{=} (\nu^D)^\tau$. Define the sequential machine $\mathfrak{M}_2 = (\mathfrak{P}(S), \nu_2, D)$ on the alphabet $\nu^\tau \cup \nu$, for which:

$$(S_1, (S_2, \sigma, S_3), (s, \sigma, s'), S_4) \in \nu_2$$

$$\Leftrightarrow (S_1, \sigma, S_4) \in \nu^{D\tau}$$

$$\wedge\ (S_2, \sigma, S_3) \in \nu^\tau$$

$$\wedge\ S_1 \cap S_3 = \{s'\}$$

$$\wedge\ S_2 \cap S_4 = \{s\}$$

$$\wedge\ (s, \sigma, s') \in \nu$$

Informally, the operation of $\mathfrak{M}_2$ can be described as follows. The input symbols of $\mathfrak{M}_2$ are edges of $\mathcal{A}^\tau$. The "tentative state transition" of $\mathfrak{M}_2$, i.e., the fragment $(S_1, (S_2, \sigma, S_3), S_4)$ of an edge of $\mathfrak{M}_2$, is contingent only on the "Σ-part" of the input symbol and is determined as an edge of $\mathcal{A}^{D\tau}$. This "tentative state transition", jointly with the input symbol, determines first, whether a transition actually takes place (edge is actually defined) in $\mathfrak{M}_2$ and second, what the output symbol, itself an edge of $\mathcal{A}$, will be.

In the definition of ν_2, the last term on the righthand side is actually superfluous since the conjunction of the other terms implies it. Indeed:

$$(S_1, \sigma, S_4) \in \nu^{D\tau} \wedge s \in S_4 \Rightarrow \exists s_0 \in S_1[(s_0, \sigma, s) \in \nu^D]$$
$$\Rightarrow \exists s_0 \in S_1[(s, \sigma, s_0) \in \nu]$$

$$s \in S_2 \wedge (s, \sigma, s_0) \in \nu \wedge (S_2, \sigma, S_3) \in \nu^\tau \Rightarrow s_0 \in S_3$$

$$s_0 = S_1 \cap S_3 \Rightarrow s_0 = s' \Rightarrow [(s, \sigma, s') \in \nu].$$

If we denote, by P_2, the sequential mapping associated with $\mathfrak{M}_2$, it follows from (4), (5) that:

$$P_{\mathcal{A}} = \rho(P_2) \circ P_{\mathcal{A}^\tau}.$$

Finally, $\mathcal{A}^\tau$ is an elementary NDA such that $\nu^\tau \subseteq (\mathfrak{P}(S) \times \Sigma) \times \mathfrak{P}(S)$ is functional and all states of $\mathcal{A}^\tau$ are distinguished as terminal, hence $P_{\mathcal{A}^\tau}$ is a sequential mapping. Taking $P_1 \overset{\text{def}}{=} P_{\mathcal{A}^\tau}$, the proof is complete. □

Note that the domain of the sequential mapping P_1 is Σ^*.

Corollary 7.9: Given F as in the theorem, there exist sequential mappings F_1, F_2 such that:

$$F = F_2 \circ \rho \circ F_1 \circ \rho (= F_2 \circ \rho(F_1)).$$

Proof: Apply the theorem to the LP FAD function $\rho \circ F \circ \rho$ to obtain:

$$\rho \circ F \circ \rho = \rho \circ F_2 \circ \rho \circ F_1$$

where F_1, F_2 are sequential mappings. Since ρ^2 is the identity:

$$F = \rho \circ (\rho \circ F \circ \rho) \circ \rho$$
$$= \rho \circ (\rho \circ F_2 \circ \rho \circ F_1) \circ \rho = F_2 \circ (\rho \circ F_1 \circ \rho)$$

as desired. □

We remark that, given a functional S-transduction $F \subseteq \Sigma^* \times \Sigma^*$, we may write by virtue of Theorems 5.1 and 7.8:

$$F = h \circ \rho(F_2) \circ F_1,$$

where F_1, F_2 are sequential mappings and h is a homomorphism. Then:

$$F = \rho(h \circ F_2) \circ F_1,$$

and $h \circ F_2$ may be interpreted as the mapping associated with a sequential machine whose input states are letters in Σ' and whose output states are words in $(\Sigma')^*$.

Returning to Theorem 7.8, we may interpret it as follows. Visualize $u \in \operatorname{dom} F$ as a finite tape with symbols from Σ written on it. The LP FAD function F is performed by two sequential machines as follows: Machine $\mathfrak{M}_1$ starts on the left end of the tape and advances, without stops or reversals, toward the right end printing, after erasing,

† The steps (2)–(4) of this proof are related to the computations made in [FCH] for checking a (one-dimensional, combinational) iterative system for the regularity property.

on the tape symbols from an augmented alphabet Σ' as it moves, thus performing sequential mapping F_1. Next, machine $\mathfrak{M}_2$ starts on the right end of the tape (which now has symbols from Σ' on it) and advances, without stops or reversals, in the opposite direction (toward the left end) printing symbols from Σ as it moves, thus performing $\rho(F_2)$, where F_2 is the sequential mapping associated with $\mathfrak{M}_2$. The resultant tape is $F(u)$.

8. Further closure properties of transductions

8.1: Generalized composition.

Given an n-ary relation R_1 and an m-ary relation R_2 (over Σ), the composite $R_2 * R_1$ is defined as the following $(n + m - 2)$-ary relation:

$$(u_1, \cdots, u_{n-1}, v_2, \cdots, v_m) \in R_2 * R_1$$
$$\Leftrightarrow \exists w[(u_1, \cdots, u_{n-1}, w) \in R_1$$
$$\wedge (w, v_2, \cdots, v_m) \in R_2].$$

We observe that given an LP relation $L \subseteq (\Sigma^k)^*$ it may be interpreted as an LP binary relation $L \subseteq (\Sigma^{k_1})^* \times (\Sigma^{k_2})^*$ (over different alphabets) for any k_1, $1 \leq k_1 \leq k - 1$, $k_1 + k_2 = k$. With this in mind, the argument of Theorem 4.13 may be extended to show that if R_1, R_2 are transductions, then $R_2 * R_1$ is a transduction.

As a special case, given functions $f : (\Sigma^*)^m \rightarrow \Sigma^*$, $g : (\Sigma^*)^n \rightarrow \Sigma^*$, the composite $h : (\Sigma^*)^{n+m-1} \rightarrow \Sigma^*$ is defined by:

$$h(x_1, \cdots x_{m-1}, y_1, \cdots y_n) = f(x_1, \cdots x_{m-1}, g(y_1, \cdots y_n))$$

where $x_1, \cdots, x_{m-1}, y_1, \cdots y_n \in \Sigma^*$. If f, g are transductions, then so is h.

As a further consequence, we have that given an m-ary transduction R over Σ and the m FAD subsets $D_1, \cdots D_m$ of Σ^*, the relation R' defined by:

$$(u_1, \cdots u_m) \in R' \Leftrightarrow (u_1, \cdots u_m) \in R$$
$$\wedge u_1 \in D_1 \wedge \cdots \wedge u_m \in D_m$$

is a transduction.

One should note, however, that the class of transductions is not closed under a more general kind of composition. Let $R_1 \subseteq (\Sigma^*)^m \times (\Sigma^*)^n$, $R_2 \subseteq (\Sigma^*)^n \times (\Sigma^*)^p$. We cannot conclude that $R_2 \circ R_1$ is a transduction. In particular, let R_1', R_2' be two binary transductions and define $R_1 = \{(u, (u, v)) : (u, v) \in R_1'\}$, $R_2 = \{((u, v), v) : (u, v) \in R_2'\}$. Then $R_2 \circ R_1 = R_2' \cap R_1'$ is, in general, not a transduction, (cf. p. 9).

8.2: Existential quantification or projection

Given the m-ary relation R, the $(m - 1)$-ary relation R' defined by

$$(u_1, \cdots u_{m-1}) \in R' \Leftrightarrow \exists u[(u_1, \cdots u_{m-1}, u) \in R]$$

is said to have been obtained from R by existential quantification. Clearly, if R is a transduction, so is R', for given an NDA $\mathfrak{A}$ that defines R, an NDA $\mathfrak{A}'$ that defines R' may be obtained by replacing all labels $(v_1, \cdots v_m)$ in $\mathfrak{A}$ by $(v_1, \cdots v_{m-1})$.

8.3: Cartesian product

Given an n-ary relation R_1 and an m-ary relation R_2 over Σ, their Cartesian product $R_2 \times R_1$ is defined by:

$$(u_1, \cdots, u_{n+m}) \in R_2 \times R_1 \Leftrightarrow (u_1, \cdots u_n) \in R_1$$
$$\wedge (u_{n+1}, \cdots u_{n+m}) \in R_2.$$

Clearly, if R_1, R_2 are transductions, so is $R_2 \times R_1$. Let NDA's $\mathfrak{A}_i'$ define the R_i, $i = 1, 2$, respectively. Replace each label $(v_1, \cdots v_n)$ of $\mathfrak{A}_1'$ by $(v_1, \cdots v_n, \underbrace{\Lambda, \cdots \Lambda}_{m})$ to yield $\mathfrak{A}_1$ and replace each label $(v_1, \cdots v_m)$ of $\mathfrak{A}_2'$ by $(\underbrace{\Lambda, \cdots \Lambda,}_{n}$ $v_1, \cdots v_m)$ to yield $\mathfrak{A}_2$. Then, $R_2 \times R_1$ is precisely $T(\mathfrak{A}_2) \cdot T(\mathfrak{A}_1)$ and hence a transduction (Proposition 3.5).

8.4: Identification of variables

Given a binary relation R, the set R' defined by

$$u \in R' \Leftrightarrow (u, u) \in R$$

is said to have been obtained from R by "identification of variables". Remark 5.5 exhibits a transduction R for which R' is not an FAD set.

Note that since R' may also be obtained by existential quantification from $R \cap D$, where D is the diagonal of $\Sigma^* \times \Sigma^*$, the class of transductions is not closed under intersection with the diagonal D.

9. Examples and counterexamples

In this section, we present examples of relations, some of which are transductions and some of which are not. In particular, examples below show that the immediate consequence relations of Post normal systems are transductions, as well as the atomic step functions of Turing machines and Markov algorithms.

9.1: The 'potential behavior' of iterative logical systems.

An "iterative system" has been defined as the collection of all finite iterative logical circuits (nets) that share a common cell and boundary conditions ([FCH], pages 3–7.) We consider the one-dimensional systems ([FCH], Fig. 5.8, page 91). Let X, S, U, W, Z be the finite sets of input, cell state, right carry, left carry and output signals respectively.

Associated with the 'typical cell' of the system $\mathcal{S}$ are the functions:

$$S_{\mathcal{S}} : X \times S \times U \times W \rightarrow S$$

$$U_\zeta : X \times S \times U \times W \rightarrow U$$

$$W_\zeta : X \times S \times U \times W \rightarrow W$$

$$Z_\zeta : X \times S \times U \times W \rightarrow Z.$$

Consider the circuit of length k in $\mathcal{S}$ and let $x_1 \cdots x_k$, $s_1 \cdots s_k$, $u_1 \cdots u_k$, $w_1 \cdots w_k$, $z_1 \cdots z_k$ denote its input, cell state, right carry, left carry and output 'arrays' respectively. Typically, x_i is intended to represent the input signal to the i^{th} cell, $1 \leq i \leq k$. (The word 'array' is intended to emphasize the spatial character of these sequences in this interpretation.) Let the relation $E \subseteq (X \times S \times U \times W \times Z)^2$ be defined by:

$$((x, s, u, w, z), (x', s', u', w', z')) \in E$$

$$\Leftrightarrow s\ S_\zeta(x, s, u, w) \wedge z = Z_\zeta(x, s, u, w) \wedge$$

$$s' = S_\zeta(x', s', u', w') \wedge z' = Z_\zeta(x', s', u', w') \wedge$$

$$u' = U_\zeta(x, s, u, w) \wedge w = W_\zeta(x', s', u', w').$$

Let the quinary LP relation E_ζ be defined by:

$$(x_1 \cdots x_k, s_1 \cdots s_k, u_1 \cdots u_k, w_1 \cdots w_k, z_1 \cdots z_k) \in E_\zeta$$

$$\Leftrightarrow \mathbf{u}(0) = [u_0 \wedge \exists t'[\forall t[t > t' \Rightarrow$$

$$\mathbf{x}(t) = \mathbf{s}(t) = \mathbf{u}(t) = \mathbf{w}(t) = \mathbf{z}(t) = \beta] \wedge$$

$$\forall t[t < t' \Rightarrow ((\mathbf{x}(t), \mathbf{s}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{z}(t)),$$

$$(\mathbf{x}(t+1), \mathbf{s}(t+1), \mathbf{u}(t+1), \mathbf{w}(t+1), \mathbf{z}(t+1))) \in E]$$

$$\wedge\ \mathbf{w}(t') = w_0],$$

where u_0, w_0 are the 'boundary conditions'. The relation E_ζ is the 'equilibrium relation' associated with the system $\mathcal{S}$, and has the following *interpretation*: If one 'applies and holds' the input array $x_1 \cdots x_k$, the circuit will operate as an 'autonomous' logical net [BW]. Ultimately, it will enter either a state cycle (of length > 1) or one of several equilibrium states (state cycles of length 1). The arrays $s_1 \cdots s_k$, $u_1 \cdots u_k$, $w_1 \cdots w_k$ jointly represent an equilibrium state under $x_1 \cdots x_k$ and $z_1 \cdots z_k$ is then the output array produced in this equilibrium state.

The (potential) equilibrium behavior of $\mathcal{S}$ is the LP relation R_ζ:

$$(x_1 \cdots x_k, z_1 \cdots z_k) \in R_\zeta \Leftrightarrow \exists s_1 \cdots s_k, u_1 \cdots u_k,$$

$$w_1 \cdots w_k[(x_1 \cdots x_k, s_1 \cdots s_k, u_1 \cdots u_k,$$

$$w_1 \cdots w_k, z_1 \cdots z_k) \in E_\zeta].$$

Clearly, both E_ζ and its projection R_ζ are FAD and, thus, transductions. Note that the 'behavior' R_ζ is 'potential' in the following sense: one may choose for a circuit of $\mathcal{S}$ an input array and an initial cell state array in such a fashion that when in 'autonomous operation under the input array', the circuit ultimately cycles and never exhibits its (equilibrium) behavior.

9.2: The 'immediate consequence relation' of a combinatorial system†

A "production"* or "rewriting rule" is an expression ([MD]), page 82:

$$f X g Y h \rightarrow p X q Y r,$$

where f, g, h, p, q, r are sequences in Σ^* and X, Y are variables over Σ^*. With such a production, one associates the following (binary) "immediate consequence relation" $R \subseteq \Sigma^* \times \Sigma^*$:

$$(u, v) \in R \Leftrightarrow \exists xy \in \Sigma^*[u = fxgyh \wedge v = pxqyr].$$

Let D_Σ denote the diagonal of $(\Sigma^*)^2$, then R is precisely $(f, p) \cdot D_\Sigma \cdot (g, q) \cdot D_\Sigma \cdot (h, r)$ and hence a transduction. Clearly, R is symmetrically locally finite.

Since $(u, v) \in R \Rightarrow l_v - l_u = (l_p + l_q + l_r) - (l_f + l_g + l_h) = C$, c constant, R is bounded. Hence, R is FAD (Theorem 6.1). If $f = g = h = \Lambda \Rightarrow p = q = r = \Lambda$, then R is an S-transduction.

Given a finite collection of such productions, the associated immediate consequence relation is the union of the immediate consequence relations associated with each rule and, thus, FAD and bounded.

One then defines R^t, the transitive closure of R, as the smallest relation that contains R and is closed under the Pierce product. One may write:

$$R^t = \bigcup_{i=1}^{\infty} R^i$$

where

$$R^i = \underbrace{R \circ R \circ \cdots R}_{i}.$$

In these terms, a combinatorial system is a finite collection of productions and a sequence $u_0 \in \Sigma^*$ called the "axiom". The "theorems" of the system are members of $\{v : (u_0, v) \in R^t\}$.

In what follows the following subset R^∞ of the transitive closure R^t of R is found useful:

$$(u, v) \in R^\infty \Leftrightarrow (u, v) \in R^t \wedge v \notin \text{dom } R^t.$$

† In the sense of [MD], p. 84.
* "Production" has been used by E. L. Post, who originated the term, in a wider sense.

9.3: Post's normal systems.

A 'normal system' [ELP] consists of an axiom $u_0 \in \Sigma^*$ and a finite collection of productions:

$$fZ \to Zr$$

where $f, r \in \Sigma^*$ and Z is a variable over Σ^*. Since such productions are special cases of those in 9.2, namely with $g = h = \Lambda$ and $p = q = \Lambda$, we may conclude that the immediate consequence relation in such a system is FAD and bounded.

9.4: The 'atomic step function' of a Turing machine.

Consider the Turing machine [MD] $\mathfrak{Z} = (Q, \bar{S}, \{R, L\}, Z)$ where Q is a finite set of internal configurations or states, $\bar{S}$ is a finite set of tape symbols, and Z a finite set of quadruples of the form: (q, S, S', q') or (q, S, R, q') or (q, S, L, q'). (The symbols R, L are interpreted as 'move right', 'move left' respectively.) An 'instantaneous description' (i.d.) of the Turing machine is a sequence:

$$S_1 \cdots S_{i-1} q \; S_i \cdots S_m$$

in $\bar{S}^* Q \bar{S} \bar{S}^*$, interpreted to mean that $\mathfrak{Z}$ is in the state q reading S_i on the tape $S_1 \cdots S_m$. The atomic step function f of $\mathfrak{Z}$ is the mapping that takes the 'i^{th} i.d.' into the '$i + 1^{st}$ i.d.' of $\mathfrak{Z}$.

We may use the results of 9.2 to show f is a transduction. Indeed, consider the following finite set of productions on the alphabet $\bar{S} \cup Q$:

1) $XqSS'Y \to XSq'S'Y$ for all $(q, S, R, q') \in Z$

2) $XS'qSY \to Xq'S'SY$ for all $(q, S, L, q') \in Z$

3) $XqSY \to Xq'S'Y$ for all $(q, S, S', q') \in Z$

4) $XYqS \to XYSq'0$ for all $(q, S, R, q') \in Z$

5) $qSXY \to q'0SXY$ for all $(q, S, L, q') \in Z$

where 0 is a special symbol in $\bar{S}$. The productions (1) and (2), which 'yield the right and left motions' of $\mathfrak{Z}$ when it is not reading end symbols, and the productions (3), which 'yield the writing action' of $\mathfrak{Z}$, are of the kind described in 9.2 with $f = h = p = r = \Lambda$. The productions (4) and (5), which 'yield the right (resp. left) motion' of $\mathfrak{Z}$ when it is reading an end symbol, are of the kind described in 9.2 with $f = g = p = q = \Lambda$ (resp. $g = h = q = r = \Lambda$).

If R is the immediate consequence relation associated with these productions, from 9.2 we have that R is a bounded transduction. Since f is the restriction of R to the FAD set $\bar{S}^* Q \bar{S} \bar{S}^*$, f is a (functional) S-transduction (Corollary 4.10).

For the atomic step function f of a Turing machine, the above conclusion may be reached more directly as follows. Consider the following subsets of $((\bar{S} \cup Q)^*)^2$:

$$\vec{M} \overset{\text{def}}{=} \{(qSS', Sq'S') : (q, S, R, q') \in Z\}$$

$$\overleftarrow{M} \overset{\text{def}}{=} \{(S'qS, q'S'S) : (q, S, L, q') \in Z\}$$

$$W \overset{\text{def}}{=} \{(qS, q'S') \quad : (q, S, S', q') \in Z\}$$

$$B \overset{\text{def}}{=} \{(qS, q'0S) \quad : (q, S, L, q') \in Z\}$$

$$E \overset{\text{def}}{=} \{(qS, Sq'0) \quad : (q, S, R, q') \in Z\}$$

$$D \overset{\text{def}}{=} \text{diagonal of } \bar{S} \times \bar{S}.$$

Then, $f = D^*(\vec{M} \cup \overleftarrow{M} \cup W)D^* \cup D^*E \cup BD^*$ is a transduction (Proposition 3.5). Since all finite relations involved contain only admissible pairs, f is an S-transduction (Corollary 3.20).

According to this notation, the computation Res_Z of $\mathfrak{Z}$ [MD, page 7] is precisely f^∞.

9.5: The atomic step function of a Markov normal algorithm

A Markov normal algorithm is a pair $(\Sigma, \mathfrak{L})$ where Σ is a finite alphabet and $\mathfrak{L}$ a finite ordered set of k 'rules' of the forms:

$$g \to q \quad \text{or} \quad g \to . \, q$$

where $g, q \in \Sigma^*$ and the dot following the arrow distinguishes among the 'rules' a subset of 'stopping rules'. With each rule is associated an atomic step function m defined by:

$$(u, v) \in m \Leftrightarrow \exists x, y \in \Sigma^*[u = xgy$$
$$\wedge \forall x'[u = x'gy \Rightarrow l_x \leq l_{x'}] \wedge v = xqy];$$

that is, v is obtained by replacing the *leftmost* occurrence of factor g in u by q.

Consider the following formula $F[u, v]$ of the language L:

$$\exists t[\mathbf{u}(0) \cdots \mathbf{u}(t-1) = \mathbf{v}(0) \cdots \mathbf{v}(t-1)$$
$$\wedge \mathbf{u}(t) \cdots \mathbf{u}(t + l_g - 1) = g$$
$$\wedge \forall t'[\mathbf{u}(t') \cdots \mathbf{u}(t' + l_g - 1) = g \Rightarrow t' \geq t]$$
$$\wedge \mathbf{v}(t) \cdots \mathbf{v}(t + l_q - 1) = q$$
$$\wedge \forall t''[t'' \geq t + l_g \Rightarrow \mathbf{u}(t'') = \mathbf{v}(t'' + l_q - l_g)]].$$

Since $(u, v) \in m \Leftrightarrow F[u, v]$, m is FAD by Theorem 10. Further, m is bounded and if $g = \Lambda \Rightarrow q = \Lambda$, then m is an S-transduction.

With a Markov algorithm there is associated a function $M \subseteq \Sigma^* \times \Sigma^*$ which may be informally described as follows. Let $(m_1, \cdots m_k)$ be the ordered set of atomic step functions associated with the k rules of the algorithm. Given $u \in \Sigma^*$, we say 'rule j is applicable to u' iff $m_j(u)$ is defined while $m_1(u), \cdots, m_{j-1}(u)$ are all undefined (for $1 \leq j \leq k$), i.e., the only rule 'applicable to u' is the first rule j for which $m_j(u)$ is defined. To obtain $M(u)$, one determines a finite sequence of words $u_0, u_1, \cdots, u_n$; $n > 0$ that obey:

1) $u_0 = u$

2) $u_{i+1} = m_j(u_i)$ where rule j is applicable to u_i, $0 \leq i \leq n-1$

3) either $u_n = m_j(u_{n-1})$ and rule j is a stopping rule, or no rule is applicable to u_n.

If there exists an infinite sequence $u_0, u_1, \cdots u_n, \cdots$ that obeys (1) and (2), $M(u)$ is not defined; otherwise $M(u) = u_n$.

Formally, we construct M from $(m_1, \cdots m_k)$ in two steps. First, define:

$$D_i \overset{\text{def}}{=} \begin{cases} \text{dom } m_i & i = 1 \\ \text{dom } m_i - \bigcup_{j=1}^{i-1} \text{dom } m_j & i > 1 \end{cases}$$

$$\xi_i \overset{\text{def}}{=} D_i \upharpoonright m_i \qquad 1 \leq i \leq k$$

that is, ξ_i is the atomic step function associated with rule i, but restricted to the domain of those sequences to which rule i is 'applicable'. In particular, the ξ_i are disjoint. Next, define:

$$\alpha \overset{\text{def}}{=} \bigcup_{\substack{\text{over the non-}\\ \text{stopping rules}}} \xi_i, \qquad \beta \overset{\text{def}}{=} \bigcup_{\substack{\text{over the}\\ \text{stopping rules}}} \xi_i$$

$$\gamma \overset{\text{def}}{=} \text{diagonal of } \left(\Sigma^*_- \bigcup_{i=1}^{k} \text{dom } m_i\right)^2.$$

Then, one obtains:

$$M = (\beta \cup \gamma) \circ \alpha^\infty \cup (\beta \cup \gamma).$$

9.6: The concatenation relation

The (ternary) relation R over Σ defined by $(u, v, w) \in R \Leftrightarrow uv = w$ is an example of a transduction that is not FAD. Let L be the LP transduction $L = F^*G^*$ where $F \overset{\text{def}}{=} \{(\sigma, \beta, \sigma) : \sigma \in \Sigma\}$ and $G = \{(\beta, \sigma, \sigma) : \sigma \in \Sigma\}$. Since $R = d(L)$, invoking Proposition 4.8, we conclude that R is a transduction.

Assume R is FAD so that by virtue of Theorem 10 there exists a formula $F[u, v, w]$ of L such that $(u, v, w) \in R \Leftrightarrow F[u, v, w]$. Let the binary relation R' be defined by:

$$(v, w) \in R' \Leftrightarrow \exists u[F[u, v, w] \wedge \forall t[u(t) = v(t)]].$$

Then R' is FAD, by virtue of the same theorem. However, $R' = \{(v, vv)\}$ which is symmetrically locally finite yet unbounded, producing a contradiction (Theorem 6.1). Hence, R is not FAD.

Note that 'unary addition' is concatenation for $\Sigma = \{1\}$, and thus a transduction, but not FAD. (The 'p-ary addition' is an FAD relation.)

Examples 9.8–9.11 give relations that are not transductions.

9.7: Modus ponens

Inasmuch as modus ponens is the immediate consequence relation of many deductive systems, it is natural to ask whether modus ponens is a transduction. The answer is, however, ambiguous depending upon exactly how one understands modus ponens. For example, let the alphabet be $\{p_1, p_2, \cdots, p_r, \rightarrow\}$, and let a well-formed formula be of the prefixed operator type so that $p_1, \rightarrow p_1p_2, \rightarrow\rightarrow p_1p_2p_3, \rightarrow p_1 \rightarrow p_2p_3$, etc., are well-formed formulas. If one understands, by modus ponens, the set of all ordered triples $(A, \rightarrow AB, B)$ where A, B are well-formed formulas, then modus ponens is not a transduction since the set of all well-formed formulas is not recognizable by a finite automaton. On the other hand, if A, B above are allowed to vary over arbitrary strings, then the resulting relation, say R, is a slight variant of concatenation and may be shown in a similar way to be transduction. The justification for understanding modus ponens in the wider sense lies in the fact that when it is applied to a pair of well-formed formulas $(A, \rightarrow AB)$, the result B is a well-formed formula.

9.8: The reversal function ρ

For an alphabet Σ of more than one symbol, let the reversal ρ be defined by:

$$\rho \overset{\text{def}}{=} \{(\sigma_1 \cdots \sigma_m, \sigma_m \cdots \sigma_1) : \sigma_1 \cdots \sigma_m \in \Sigma^*\}$$

and T be the FAD function $T \overset{\text{def}}{=} \{(\sigma_0, \sigma_0)\}^*(\sigma_1, \sigma_1)\cdot\{(\sigma_0, \sigma_0)\}^*$ for some pair $\sigma_0 \neq \sigma_1$; $\sigma_0, \sigma_1 \in \Sigma$. Then, $\rho \cap T = \{(\sigma_0, \sigma_0)^n(\sigma_1, \sigma_1)(\sigma_0, \sigma_0)^n : n \geq 0\}$. Since $\rho \cap T$ is not FAD, neither is ρ. But ρ is LP and, using Corollary 6.6, we conclude ρ is not a transduction.

9.9: Multiplication, unary and binary

Consider first the 'unary multiplication relation' R over $\Sigma = \{1\}$ defined by:

$$R \overset{\text{def}}{=} \{(1^m, 1^n, 1^{mn}) : m, n \geq 0\}.$$

If R is a transduction, then the relation R' defined as:

$$(v, w) \in R' \Leftrightarrow \exists u[(u, v, w) \in R]$$

must be a transduction as well.

Next, consider the 'binary multiplication relation' S over $\Sigma = \{0, 1\}$, defined as follows: $(u, v, w) \in S$ iff w is a binary representation of the product of the numbers whose binary representations are u, v. For purposes of this definition, a string in Σ^* will be a binary number with its rightmost digit least significant. [We could restrict the relation to strings in $1 \cdot \Sigma^*$, i.e., binary numbers whose most significant digit is 1 without changing the discussion to follow.] Thus, for example, both (11, 0110, 10010), (0011, 110, 010010) $\in S$.

If S is a transduction, then the relation S' defined as:

$$(v, w) \in S' \Leftrightarrow \exists u[(u, v, w) \in S] \wedge v, w \in \{1\}^*$$

must be a transduction as well (8.1, 8.2).

It is, however, readily verified that both R', S' are identical to the following relation $K \subseteq \{1\}^* \times \{1\}^*$:

$(u, v) \in K \Leftrightarrow v \in \{u\}^*.$

It thus remains to establish that K is not a transduction. (Remark: the question whether the diadic multiplication relation is a transduction may also be reduced to the question whether K is.)

Assume, to the contrary, that K is defined by some NDA $\mathcal{A}$. Clearly, one may assume that all labels in $\mathcal{A}$ are of lengths ≤ 1, i.e., of the forms $(\Lambda, \Lambda), (\Lambda, 1), (1, \Lambda), (1, 1)$. Call a path in $\mathcal{A}$ a 'simple inadmissible loop' iff (1) there exists a state s in $\mathcal{A}$ such that the path connects s to itself without passing through s, and (2) the label of the path is inadmissible. Let the number of states of $\mathcal{A}$ be n. Since $(1, 1^{2n+1}) \in K$, there exists a subpath of a successful path in $\mathcal{A}$ of length at least n whose label is inadmissible and hence an SI (simple inadmissible) loop. Now the number of SI loops in $\mathcal{A}$ is finite. Let $M \geq 2$ be the smallest integer that exceeds the length of any label of a SI loop in $\mathcal{A}$. Consider the pair $(1^M, 1^{2Mn})$. Since this pair is in K, it is the label of a successful path in $\mathcal{A}$.

This path contains only M edges with labels $(1, u)$ and hence at least one subpath of length at least n with an inadmissible label. Consequently, the path contains a SI loop as a subpath. Let the label of this SI loop be $(\Lambda, 1^m)$. Then, $(1^M, 1^{2Mn+m})$ is in K; that is, M divides $2Mn + m$. Finally, M divides m, which, since $m < M$, is a contradiction. We have, then, that K is not a transduction and hence neither are the multiplication relations R, S.

9.10: $v \in \{u\}^*$

It has been shown above that K is not a transduction. As a corollary, by 8.1, the relation $R = \{(u, v) : v \in \{u\}^*\}$ over an arbitrary alphabet Σ is not a transduction.

9.11: The homomorphism γ_0 of $F_{\Sigma \cup \Sigma'}$ onto the representatives of the free group on $\Sigma \cup \Sigma'$

Let Σ, Σ' be disjoint finite alphabets in a 1 : 1 correspondence with one another, i.e., σ' is the 'formal inverse' of σ for each $\sigma' \in \Sigma'$. Define $A \stackrel{\text{def}}{=} \{(\sigma\sigma', \Lambda) : \sigma \in \Sigma\}$, $B \stackrel{\text{def}}{=} \{(\sigma'\sigma, \Lambda) : \sigma \in \Sigma\}$ and $D \stackrel{\text{def}}{=}$ diagonal of $(\Sigma \cup \Sigma')^2$; and let:

$$\tau \stackrel{\text{def}}{=} D^*(A \cup B)D^*.$$

Then, $\gamma_0 = \tau^\infty \cup \{\Lambda\}$.

Consider the set $\gamma_0^c(\Lambda)$. A sequence in $(\Sigma \cup \Sigma')^*$ is in $\gamma_0^c(\Lambda)$ iff it may be 'reduced to Λ by repeated cancellations of inverse pairs $\sigma\sigma'$, $\sigma'\sigma$.' Given some $\sigma \in \Sigma$, $\gamma_0^c(\Lambda) \cap \{\sigma\}^*\{\sigma'\}^* = \{\sigma^n(\sigma')^n : n \geq 0\}$ is not FAD and hence neither is $\gamma_0^c(\Lambda)$. Thus, γ_0 is not a transduction.

10. Appendix: the language L

Let β be a symbol $\notin \Sigma$ and N stand for the set of non-negative integers. We shall use **u**, **v** as variables that take on values in $(\Sigma \cup \{\beta\})^N$, i.e., infinite sequences over Σ_β, with the property P: $\exists\, m \forall t \geq m\, [\mathbf{u}(t) = \beta]$. The mapping $\mathbf{u} \rightarrow u$ which takes **u** into $\mathbf{u}(0)\mathbf{u}(1) \cdots \mathbf{u}(m_s - 1)$, where m_s is the smallest m for which $\forall t \geq m\, [\mathbf{u}(t) = \beta]$, is a 1 : 1 correspondence between infinite sequences over Σ_β with the property P and strings in $(\Sigma_\beta)^*$ which do not terminate with β. Call this set π. We note that if $\forall t[u(t) = \beta]$, then $u = \Lambda$

$$\Sigma^* \subseteq \Pi = (\Sigma_\beta^* \cdot \Sigma u\{\Lambda\}).$$

Consider the following class L_n of interpreted formulae. The constants are $\sigma_1, \cdots \sigma_n \in \Sigma$, β, and the numerals $0, 1, \cdots$. The *individual* variables, ranging over N, are t_1, t_2, etc. The *function* variables $\mathbf{u}_1$, $\mathbf{u}_2$, etc. range over elements of $(\Sigma_\beta)^N$ with the property P. The following are atomic formulae:

1) $\mathbf{u}_i(t_j + m) = \sigma_k$ or $\mathbf{u}_i(t_j + m) = \beta$ [Note that $\mathbf{u}(t_1 + t_2)$ is not a formula of the system.]
2) $\mathbf{u}(t_i + m) = \mathbf{v}(t_j + p)$
3) $t_i + m \leq t_j + p$ or $t_i + m < t_j + p$, where m, p are numerals.

The language L_n is constructed from such atomic formulae by means of truth functional connectives ($\wedge, \vee, \sim$, etc.) and quantification of both the individual and the function variables. Thus, $\exists t$, $\forall t$, $\exists \mathbf{u}$, $\forall \mathbf{u}$ *are* permitted.

Theorem 10: If $R \subseteq (\pi)^n$ is FAD, *then there exists a formula F of L with no free individual variables such that $u_1, \cdots u_n) \in R \Leftrightarrow F[\mathbf{u}_1, \cdots \mathbf{u}_n]$ is valid. Conversely, if R is defined by the above equivalence, then R, $R \subseteq (\pi)^n$ is* FAD.

In describing the language L, we have not attempted to use only a minimal set of primitives but rather have chosen merely a convenient set. For example, the atomic formula "$t_1 < t_2$" is dispensable in the sense of being definable in terms of the other primitives. On the other hand, we have not included all atomic formulas which we actually use in the examples as atomic formulas of L, leaving it to the reader to see that the actual formula written may be replaced by a semantically equivalent one in L. In order to avoid confusion, we explain two such cases.

1) $\mathbf{u}(t)\mathbf{u}(t + 1) \cdots \mathbf{u}(t + l_g - 1) = g$, where $g = \sigma_0\sigma_1 \cdots \sigma_{l(g)-1}$ and l_g is a numeral, is equivalent to
1') $\mathbf{u}(t) = \sigma_0 \wedge \mathbf{u}(t + 1) = \sigma_1 \wedge \cdots \mathbf{u}(t + l_g - 1) = \sigma_{l(g)-1}$
2) $\mathbf{u}(0) \cdots \mathbf{u}(t - 1) = \mathbf{v}(0) \cdots \mathbf{v}(t - 1)$ is equivalent to
2') $(\forall x)\,(x < t \rightarrow \mathbf{u}(x) = \mathbf{v}(x))$.

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References

[BW] A. W. Burks and J. B. Wright, "Theory of Logical Nets", *Proc. IRE* **41**, 1357 (1953).

[CCE] C. C. Elgot, "Decision Problems of Finite Automata Design and Related Arithmetics", *Trans. Amer. Math. Soc.* 98, 21–51 (1961).

[ELP] E. L. Post, "Formal Reductions of the General Combinatorial Decision Problem", *Amer. Journal of Math.* **65**, 197–215 (1943).

[EM] C. C. Elgot and J. E. Mezei, "Two-sided Finite-state Transductions" (an earlier version of the present paper), IBM Research Report *RC 1017*, June, 1963.

[FCH] F. C. Hennie, *Iterative Arrays of Logical Circuits*, M. I. T. Press, 1961.

[MD] M. Davis, *Computability and Unsolvability*, McGraw-Hill, 1958.

[MPS] M. P. Schützenberger, "A Remark on Finite Transducers", *Information and Control* **4**, 185–196 (1961).

[NC] N. Chomsky, "Formal Properties of Grammars" (mimeographed notes).

[RS] M. O. Rabin and D. Scott, "Finite Automata and Their Decision Problems", *IBM Journal Res. and Dev.* **3**, 114–125 (1959).

[SG] S. Ginsburg, "Some Remarks on Abstract Machines", *Trans. Amer. Math. Society* **96**, 400–444 (1960).

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