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Derivation of Maximal Compatibles Using Boolean Algebra

Given an incompletely specified flow table for a sequential switching function, it is desired to minimize the number of rows in that flow table, i.e., to find an equivalent minimum-row flow table. Paull and Unger[1] have discussed this problem in detail.

The steps in the minimization procedure can be briefly summarized as follows:

(a) All row-pairs are determined to be compatibles or incompatibles. (A pair of rows that can be equivalent to a single row in a new table is a compatible; a pair of rows that cannot be equivalent to a single row is an incompatible.) The implication table is useful in the systematic determination of row-pair compatibility.

(b) All *maximal compatibles* are derived. (Sets of more than two rows can also be compatibles; a set of rows is a compatible if every pair of rows in the set is a compatible. A maximal compatible is one that is not included in any larger compatible.)

(c) A minimal closed set of compatibles is selected, which yields the minimum-row flow table.

Some step-by-step processes for obtaining the maximal compatibles are given by Paull and Unger. A simple method using Boolean algebra is given here:

(1) For every incompatible row-pair, form the Boolean sum of the corresponding row designations, and write the Boolean product of all of these sums.

(2) Multiply out this expression to obtain an equivalent sum of products, eliminating all redundancy in the process.

(3) For each resultant product, write the set of all missing row designations; these sets constitute all of the maximal compatibles.

The method will be illustrated using an example from Paull and Unger. The implication table from that example is reproduced in Fig. 1 showing only the X entries which denote the incompatible row-pairs.

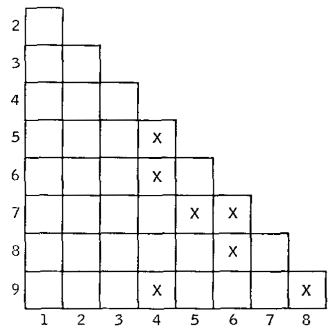

Figure 1 **Implication table with X entries showing incompatible row-pairs.**

(1) $(4+5)(4+6)(4+9)(5+7)(6+7)(6+8)(8+9)$.

(2) $= 4568 + 4679 + 478 + 569$.

(3) MC's : (12379) (12358) (123569) (123478).

In writing the initial product of sums, the row-pairs can be grouped by column, generally effecting some reduction in writing; e.g., expression (1) above could have been written

$(4+569)(5+7)(6+78)(8+9)$.

The proof of this method is as follows. Assume that m and n constitute one of the incompatible row-pairs. The product of sums will therefore include the sum $(m + n)$ in an expression of the form

$$(\ldots + \ldots)(\ldots + \ldots)(m + n)(\ldots + \ldots)\ldots(\ldots + \ldots).$$

When this expression is multiplied out, every resultant product must contain an m or an n. Therefore no set of missing row designations can contain both m and n.

Further assume that p and q constitute one of the compatible row-pairs. The product of sums will therefore not include the sum $(p + q)$. When the product of sums is multiplied out, there must be at least one resultant product with neither a p nor a q. Therefore, there must be a set of missing row designations that contains both p and q.

Thus, no set of missing row designations can contain an incompatible, but every compatible must be contained among the sets.

Lastly, if in obtaining the sum of products all redundancy is eliminated, minimal products will result, minimal products producing maximal compatibles.

The maximal incompatibles can be found by this method also: by starting with the sums corresponding to every compatible row-pair, the derived sets constitute all of the maximal incompatibles. (The number of elements in the largest maximal incompatible is a lower bound on the number of rows in the reduced flow table.[1])

This method was suggested by a paper by J. Weissman.[2]

Acknowledgments

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References

1. M. C. Paull and S. H. Unger, "Minimizing the Number of States in Incompletely Specified Sequential Switching Functions," *IRE Transactions on Electronic Computers* **EC-8**, No. 3, 356-367 (1959).
2. J. Weissman, "Boolean Algebra, Map Coloring, and Interconnecting," *The American Mathematical Monthly* **69**, No. 7, 608-613 (1962).

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