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Nonlinear Wave Propagation in a Transmission Line Loaded with Thin Permalloy Films

Abstract: This paper considers the propagation of waves in a transmission line loaded with thin permalloy films. Since the films are driven to saturation, the transmission line equations which are derived to describe the wave propagation are nonlinear. The nonlinearity requires the use of shock wave analyses, and a derivation of the appropriate shock relations is included. The problems of a line loaded with a single film and a line loaded with a periodic array of films are both treated. The saturation front moving along a string of films is shown to be the shock front. The shock speed, which is determined in terms of the parameters of the circuit, is then the operational speed limit of a thin film memory.

1. Introduction

The purpose of the present paper is to study the propagation of an electric pulse along a transmission line with a nonlinear relationship between current and magnetic flux. Such a nonlinearity can be realized by inserting a layer of ferromagnetic material between the two conductors of the line. The main feature is the saturation of the ferromagnetic material at high currents; this causes the specific inductance L of the transmission line to drop for currents above some critical value I_c to the value that is characteristic of the line without the ferromagnetic material.

The ferromagnetic layer is assumed to be an insulator or a metal strip thin enough to carry only negligible currents. The specific capacitance C is then independent of current. The propagation speed $1/\sqrt{LC}$ is large above the critical current I_c, so that shock waves will form. The high currents which have started later will tend to overtake the lower currents which have had an earlier start at the generator. In the ideal case (without losses) a continuously rising current at the generator is deformed into a discontinuously rising pulse by being propagated through this transmission line. The occurrence of shock waves in transmission lines has been observed by others. See, for example, the paper by R. Landauer[1] and its bibliography. Landauer considered the possibility of a nonlinear specific capacity giving rise to parametric amplification. Additionally, he discussed a number of questions dealing with electromagnetic shock waves in general.

The equations of the transmission line are derived in Section 2 under the assumption of no losses. It is possible to replace the one continuous permalloy film by a sequence of small permalloy thin film bits. If the length of each bit along the transmission line together with the distance of separation between bits does not exceed, say 1 mm, it is reasonable to expect that a pulse with a nanosecond rise time does not "see" this periodic structure, but only a smeared-out effect. The Appendix has the purpose of showing the equivalence between the transmission line with a continuous layer of ferromagnetic material and the transmission line with a large sequence of small bits.

The derivations in Sections 2 and the Appendix are made assuming currents and voltages that have a sufficiently continuous behavior to allow an arbitrary number of differentiations, both with respect to time and with respect to the spatial coordinate. Since the nonlinear inductance is expected to yield discontinuities in current and voltage, the differential equations of the transmission lines are supplemented in Section 3 by special conditions which have to be satisfied across the shock (discontinuity). These shock conditions follow from the physical requirements of conservation of electric charge and magnetic flux. The derivation of these relations by Landauer,[1] on the other hand, employed the transmission line equations as a starting point.

The consequences of the shock conditions and, in particular, the motion of the shock over the transmission line are analyzed in Section 4. In this Section a functional equation of motion for the shock is obtained. After this equation is appropriately specialized, situations which lead to straight-line shocks are discussed. Finally, the solution of the problem of saturating and then unsaturating the line is obtained for this case of a straight-line shock.

A brief discussion of the loss mechanisms is now given for the interested reader.

The various loss mechanisms limit a successful investigation of the nonlinear transmission line. In the present case of a thin permalloy film between two conducting metal strips, three important types of losses must be neglected:

1. Ohmic resistance in the conductors. This can be overcome by choosing a large enough cross-sectional area.

2. Eddy-current losses in the conductors increase with frequency, but they can be reduced if it is possible to divide the conductors into thin threads, so that currents are capable of flowing only in one direction.

3. The rotation of the magnetization in the thin permalloy films is damped by various mechanisms, such as spin wave excitation, which cannot easily be shifted to higher and higher frequencies. In the present setup, these losses start to be noticeable if the magnetization is turned by 90° in less than a nanosecond. Practically, this means that it is not possible with present-day techniques to sharpen the rise time of a pulse along the nonlinear transmission line discussed in this report beyond the nanosecond, while keeping the simplifying assumptions made throughout the report. On the other hand, 1 nanosecond corresponds to 30 cm (1 ft) at the speed of light *in vacuo*, and this distance is halved by the dielectric constant (~4) of the insulator between the two conductors. The effects predicted by this theory can therefore be realized in physical models of reasonable size.

2. Equations of the transmission line

In this Section the nonlinear transmission line equations are derived. The derivation utilizes the laws of Faraday and Ampère, i.e. Maxwell's equation, and a special relation for the magnetic energy of a thin film. The reader who is not interested in the details of the derivation may consult Eqs. (2.9) to (2.11) as a summary of this Section.

The transmission line is described in a Cartesian coordinate system. The plane of the thin permalloy film is the xy plane. The conductors allow the current to flow in the x direction only. The magnetization in the film,

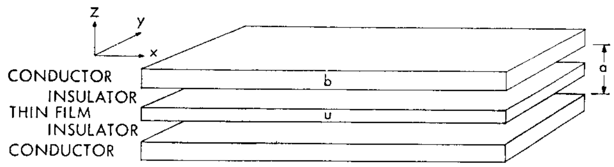


Figure 1 **Schematic of strip line loaded with thin magnetic film.**

as well as the current distribution in the conductors, is assumed to be independent of the y coordinate. This hypothesis is reasonable as long as the width of the permalloy film and of the conductors (in the y direction) is large compared to their thickness (in the z direction), as indicated in Fig. 1. The following formula for the voltage V across the line may be obtained from Faraday's law of induction in a standard manner.

$$\frac{\partial V}{\partial x} = \frac{1}{C'} \frac{\partial}{\partial t} (aH_y + 4\pi u M_y). \tag{2.1}$$

To obtain this formula, ohmic resistance in the conductor is neglected and it is supposed that neither the magnetic field **H** between the conductors nor the magnetization **M** inside the thin film depends on the coordinate z. C' is the velocity of light, since cgs units are being used. The relation between voltage and current density j_x is given by

$$\frac{\epsilon}{a} \frac{\partial V}{\partial t} = -4\pi b \frac{\partial j_x}{\partial x}, \tag{2.2}$$

where ϵ is the dielectric constant of the medium between the conductors.

If we assume that **H** has only a nonvanishing component in the y direction between the two conductors, it follows from Ampère's law that

$$H_y = \frac{4\pi}{C'} b j_x. \tag{2.3}$$

In both (2.2) and (2.3) j_x was assumed independent of z in order to integrate j_x over the thickness b of the conductor. This assumption is not necessary, and it would be legitimate to replace in both (2.2) and (2.3) the product bj_x by the integral $\int j_x dz$ over the conductor. The only reason for j_x to depend on z would be the skin-effect, which has a complicated frequency dependence. But since the voltage drop along the conductor due to ohmic resistance is neglected anyway, the skin effect can also be neglected. The integral $\int i_x dz$ is all that occurs in the present theory.

A fourth relation is necessary in order to eliminate H_y and M_y altogether from the transmission line equation. This last relation expresses the physical characteristics of the thin film. These are not determined by general laws, to be satisfied in any case, such as Maxwell's equations. The physical properties of the thin permalloy film have to be obtained from experiment.

The magnetization $\mathbf{M}$ has at each point in the film the same absolute value, constant in time. Because div $(\mathbf{H} + 4\pi\mathbf{M}) = 0$, this magnetization can be assumed to lie in the plane of the film. If this were not true, large magnetic poles would be generated on either surface of the film; these would create a magnetic field which would tend to turn the magnetization into the plane of the film. It is then sufficient to describe the state of magnetization at each point by an angle ϕ. The energy density F due to magnetization of the film is composed of two parts: the uniaxial anisotropy which tends to turn the vector $\mathbf{M}$ into one of the two "easy" directions, and the energy of $\mathbf{M}$ in the magnetic field $\mathbf{H}$, which is simply given by their scalar product. If the easy axis coincides with the x axis, and if ϕ is the angle of $\mathbf{M}$ with the x axis, F is given by

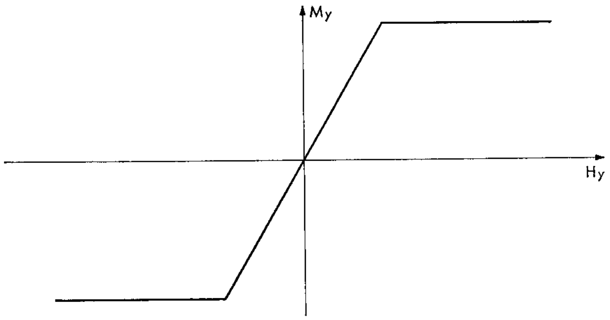
$$F = K \sin^2 \phi - |M| \cdot (H_x \cos \phi + H_y \sin \phi), \quad (2.4)$$

where K is a material constant.[2] For a given magnetic field $\mathbf{H}$ the angle ϕ is found from the minimum of F with respect to variations in ϕ, i.e. from $\partial F/\partial \phi = 0$. In the transmission line described in this paper, H_x is assumed to vanish. It follows that

$$M_y = \begin{cases} |M| \sin \phi = \dfrac{|M|^2}{2K} H_y, & \text{for } H_y \leq \dfrac{2K}{|M|} = H_c; \\ \pm |M|, & \text{for } H_y > \dfrac{2K}{|M|} \text{ or } H_y < -\dfrac{2K}{|M|}. \end{cases} \quad (2.5)$$

The equations express the "saturation" of the magnetization at magnetic fields in the y direction above the absolute value $2K/|M|$. The resulting nonlinear relation between M_y and H_y is shown in Fig. 2. Moreover, it is of practical importance to realize the steepness of the M_y versus H_y slope in the center part. Indeed, the coefficient in (2.5) has generally a value in the order of 200.

Figure 2 **Relation between magnetization and magnetic field in the hard direction of a saturable thin film.**



The physical processes which govern the relation between M_y and H_y are not reversible in the simple case of just one field component H_y, which was outlined above. If one starts at $t = 0$ with $\phi = 0$ and $H_y = 0$, and then lets H_y increase monotonically beyond H_c, M_y runs along the curve of Fig. 2. But if H_y is then allowed to decrease below H_c, without the presence of a small field H_x (much smaller in absolute value than H_c) the magnetization does not "know" where to turn, to $\phi = 0$ or to $\phi = \pi$. In general, the thin film will break up into domains, some tending to $\phi = 0$, others to $\phi = \pi$. Hence, there is the necessity of a small field component H_x, called the "word" field in opposition to the large "bit" field H_y, to guarantee the reversibility of the curve of Fig. 2. The word field will be neglected in the future, because it does not influence the propagation of pulses along the transmission line. Equation (2.1) now becomes, with the help of (2.3) and (2.5),

$$\frac{\partial V}{\partial x} = -\bar{L}(I) \cdot \frac{\partial I}{\partial t}, \quad (2.6)$$

with

$$\bar{L}(I) = \begin{cases} \dfrac{4\pi}{C'^2} \cdot \dfrac{a}{h} \left(1 + \dfrac{2\pi |M|^2}{K} \cdot \dfrac{u}{a}\right), \\ \quad \text{for } |I| = |bhj_x| \leq \dfrac{Kh}{2\pi |M|} = I_c; \\ \dfrac{4\pi}{C'^2} \cdot \dfrac{a}{h}, \quad \text{for } |I| > I_c; \end{cases} \quad (2.7)$$

and Eq. (2.2) is now written in terms of the total current $I = bhj_x$, where h is the width of the conductor, as

$$\frac{\partial I}{\partial x} = \Gamma \frac{\partial V}{\partial t} \quad \text{with} \quad \Gamma = \frac{\epsilon}{4\pi} \cdot \frac{h}{a}. \quad (2.8)$$

Instead of electrostatic cgs units used so far, the voltage is now expressed in volts, the current in amperes, the magnetization M and the critical field $H_c = 2K/|M|$ in gauss. With the permeability given by

$$\mu(I) = \begin{cases} 1 + 4\pi \cdot \dfrac{|M|}{H_c} \cdot \dfrac{u}{a}, & \text{for } |I| \leq I_c = \dfrac{10}{4\pi} \cdot hH_c; \\ 1, & \text{for } |I| > I_c. \end{cases} \quad (2.9)$$

With the capacity C per cm and the inductance L per cm given by

$$C = \frac{\epsilon h}{aZ_0C'}, \qquad L(I) = \mu(I)\cdot\frac{aZ_0}{hC'}, \tag{2.10}$$

where Z_0 is the impedance of the vacuum, 370 ohms, the equations of the transmission line reduce to

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}, \qquad \frac{\partial V}{\partial x} = -L(I)\frac{\partial I}{\partial t}. \tag{2.11}$$

This last form will be used henceforth. In the remainder of this paper, since only scalar quantities are employed, the simple subscript notation for partial derivatives will be employed.

3. Shock equations

The mathematical problem at hand is to solve the transmission line equations with a nonlinear inductance. Since the speed of the waves arising as solutions of these equations is $1/\sqrt{CL(I)}$, there will be waves of differing speeds propagating in the same line. Since the faster waves will tend to overtake slower waves and since the solution to the differential equations cannot be multiple-valued, there must be a mechanism for keeping separate the waves of differing speeds. In the parlance of mechanics this mechanism is known as a *shock wave*. If the existence of a shock is admitted into our problem, the laws of interaction of waves on opposite sides of the shock must be known.

Figure 3 **Section of strip-line containing shock front indicated by dotted line.**

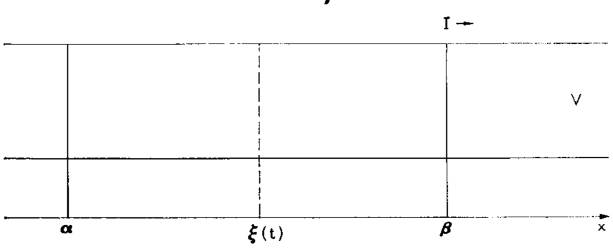
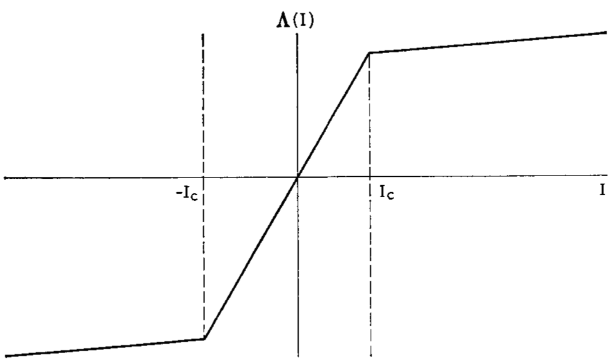


Figure 4 **Relation of flux versus current in saturable film** *(cf. Figure 2).*



This information is supplied by relations valid across a shock, the so-called *shock relations*. In mechanical problems these relations are expressions of the conservation of mass, momentum, energy, and possibly entropy across the shock wave (see Ref. 3). In the present case of electrical waves there are only two independent shock relations. These are expressions of the principles of conservation of charge and magnetic flux. These relations are derived in the following paragraphs.

The net current flow into the length $\alpha\beta$ of the line (Fig. 3) is

$$I(\alpha) - I(\beta) = \frac{dQ}{dt} = \frac{d}{dt}\int_\alpha^\beta CV\,dx$$
$$= \left(\int_\alpha^{\xi^-} + \int_{\xi^+}^\beta\right)\frac{d(CV)dx}{dt} - \dot{\xi}[CV]. \tag{3.1}$$

Here the brackets [ ] refer to the jump in the indicated quantity across $\xi(t)$, and a dot indicates time differentiation. Now let $\alpha \to \xi^-$ and $\beta \to \xi^+$. The integrals will vanish, and there remains the first shock relation

$$[I] = \dot{\xi}[CV]. \tag{3.2}$$

Let $B = H + 4\pi M$ and let $\Lambda(I)$ be the flux through unit length of line. Then

$$\int_\alpha^\beta \Lambda(I)\,dx = \iint B\,dS. \tag{3.3}$$

Here the integral of B is taken over the area of the line contained between α and β. Now

$$\frac{d}{dt}\iint B\,dS = V(\alpha) - V(\beta) \tag{3.4}$$

while

$$\frac{d}{dt}\int_\alpha^\beta \Gamma(I)\,dx = \left(\int_\alpha^{\xi^-} + \int_{\xi^+}^\beta\right)\frac{d\Gamma}{dt}\,dx - \dot{\xi}[\Gamma(I)]. \tag{3.5}$$

Letting $\alpha \to \xi^-$ and $\beta \to \xi^+$, these equations yield the second shock relation

$$[V] = \dot{\xi}[\Gamma(I)]. \tag{3.6}$$

For a thin magnetic strip in a line, a typical $\Lambda(I)$ curve is as indicated in Fig. 4, where I_c is the critical current at which the film saturates. If the specific inductance is defined to be the slope of the $\Lambda(I)$ curve, then the specific inductance will be

$$L(I) = \begin{cases} L & |I| < |I_c| \\ L_1 & |I| > |I_c| \end{cases} \tag{3.7}$$

with $L > L_1$.

In this case an easy computation gives

$$[\Gamma(I)] = [(I - I_c)L(I)]. \tag{3.8}$$

Thus (3.7) becomes

$$[V] = \dot{\xi}[(I - I_c)L(I)]. \tag{3.9}$$

For future reference note the following: If ahead of the shock ($x > \xi(t)$, $\dot{\xi} > 0$), $I < I_c$, so that $L(I) = L$ there, while behind the shock ($x < \xi(t)$), $I > I_c$, so that $L(I) = L_1$, and if V_+, I_+ and V_-, I_- denote the values of V and I at the shock (ahead and behind, respectively) as indicated in Fig. 5, then the shock relations (3.3) and (3.10) may be solved for V_- and I_-,

$$\begin{aligned} I_- &= \alpha(\dot{\xi})I_+ + \beta(\dot{\xi})I_c \\ V_- &= V_+ - \gamma(\dot{\xi})(I_+ - I_c) \end{aligned} \tag{3.10}$$

where

$$\alpha(\dot{\xi}) = \frac{1 - CL\dot{\xi}^2}{1 - CL_1\dot{\xi}^2}$$

$$\beta(\dot{\xi}) = \frac{C\dot{\xi}^2(L - L_1)}{1 - CL_1\dot{\xi}^2}$$

$$\gamma(\dot{\xi}) = \frac{\dot{\xi}(L - L_1)}{1 - CL_1\dot{\xi}^2}. \tag{3.11}$$

4. Analysis of the mathematical model

• *The equation of motion of the shock front in a semi-infinite line.*

At time $t = 0$ the transmission line is at rest, i.e., $I = V = 0$. Then the generator $\phi(t)$ is switched on causing voltage and current waves to propagate down the line. The solution to the transmission line equations is then easily seen to be

$$I(x, t) = \begin{cases} 0 , & t < \sqrt{LC}\,x \\ \dfrac{\phi(t - \sqrt{LC}\,x)}{R + \sqrt{\dfrac{L}{C}}}, & t > \sqrt{LC}\,x \end{cases} \tag{4.1}$$

$$V(x, t) = \sqrt{\frac{L}{C}}\, I(x, t). \tag{4.2}$$

(4.1) and (4.2) represent waves of speed $1/\sqrt{LC}$ (the slow waves) moving down the transmission line.

At a certain time t_1 the generator reaches and then exceeds a value of voltage [$\phi = (R + \sqrt{L/C})I_c$], which causes $I(x, t)$, as given in (4.1), to exceed the critical current I_c. Now waves of speed $1/\sqrt{L_1C}$ propagate down the line, catching up to the slow waves which are ahead. Between the two groups of waves there is a shock wave whose equation of motion will now be derived.

Figure 6 contains a sketch of the shock and two characteristics $\overline{AP}$ and $\overline{AQ}$ representing waves of speed $1/\sqrt{L_1C}$.

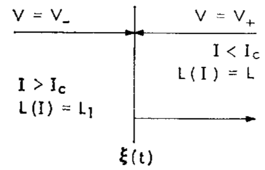


Figure 5 **Values assigned to electrical parameters on opposite sides of shock front.**

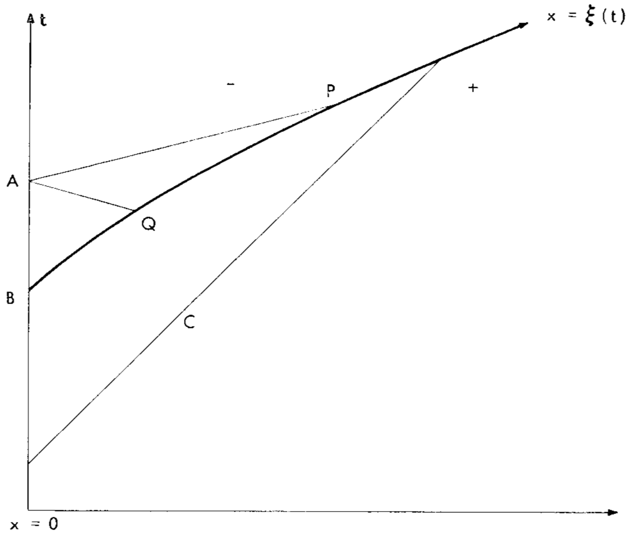


Figure 6 **Configuration showing relation between shock wave and electromagnetic waves of different speeds.**

At the point A there is an impedance boundary condition

$$V(A) + RI(A) = \phi(A). \tag{4.3}$$

At P and Q the shock relations (written in the form (3.11)) are respectively

$$I(P^-) = \alpha(P)I(P^+) + \beta(P)I_c \tag{4.4}$$

$$V(P^-) = V(P^+) - \gamma(P)(I(P^+) - I_c) \tag{4.5}$$

$$I(Q^-) = \alpha(Q)I(Q^+) + \beta(Q)I_c \tag{4.6}$$

$$V(Q^-) = V(Q^+) - \gamma(Q)(I(Q^+) - I_c). \tag{4.7}$$

Here $\alpha(P)$, for example, is $\alpha(\dot{\xi}$ (the t coordinate of P)), and P^- and P^+, for example, refer respectively to the point P behind and ahead of the shock. To these five equations are adjoined two more, which relate V and I at A to V and I at P^- and Q^-, respectively,

$$I(P^-) - I(A) = -\sqrt{\frac{C}{L_1}}\,(V(P^-) - V(A)) \tag{4.8}$$

$$I(Q^-) - I(A) = -\sqrt{\frac{C}{L_1}}\,(V(Q^-) - V(A)), \tag{4.9}$$

(These equations, which are referred to as the characteristic equations, are an expression of Ohm's law in a transmission line.)

The seven equations (4.3) to (4.9) involve the six unknowns V and I at A, P^-, and Q^- in a linear fashion, as well as the unknown $\dot{\xi}$ in a nonlinear fashion. The coefficients of these equations depend on V and I at P^+ and Q^+ (i.e., ahead of the shock). V and I ahead of the shock are given in (4.1) and (4.2). However, the x and t coordinates of P and Q are dependent on ξ itself. Solving (4.4) to (4.9) for $V(A)$ and $I(A)$ and then inserting these expressions into (4.3) gives an equation for $\dot{\xi}$.

$$\begin{aligned}2\phi(A) = {} & I(P^+)\left[\left(R + \sqrt{\frac{L_1}{C}}\right)\alpha(P) - \left(1 + R\sqrt{\frac{C}{L_1}}\right)\gamma(P)\right] + V(P^+)\left[1 + R\sqrt{\frac{C}{L_1}}\right] \\ & + I(Q^+)\left[\left(R - \sqrt{\frac{L_1}{C}}\right)\alpha(Q) - \left(1 - R\sqrt{\frac{C}{L_1}}\right)\gamma(Q)\right] \\ & + V(Q^+)\left[1 - R\sqrt{\frac{C}{L_1}}\right] + I_c\left[\left(R + \sqrt{\frac{L_1}{C}}\right)\beta(P) \right. \\ & + \left(1 + R\sqrt{\frac{C}{L_1}}\right)\gamma(P) + \left(R - \sqrt{\frac{L_1}{C}}\right)\beta(Q) \\ & \left. + \left(1 - R\sqrt{\frac{C}{L_1}}\right)\gamma(Q)\right]. \qquad (4.10)\end{aligned}$$

- *The case of matching the generator*

The equation of shock motion (4.10) simplifies if the generator is matched to the saturated line, i.e.

$$R = \sqrt{L_1/C}. \qquad (4.11)$$

This will cause waves which arrive at the left end of the line to be absorbed, thus simplifying the analysis. This choice of R will be adhered to in the remainder of this report. In this case, (4.10) becomes, upon using the values of α, β, and γ as given in (3.11),

$$\begin{aligned}\phi(A) = {} & \frac{R(1 - CL\dot{\xi}^2) - \dot{\xi}(L - L_1)}{1 - CL_1\dot{\xi}^2}\, I(P^+) + V(P^+) \\ & + I_c\left[(L - L_1)\,\frac{\dot{\xi} + RC\dot{\xi}^2}{1 - CL_1\dot{\xi}^2}\right]. \qquad (4.12)\end{aligned}$$

- *Legitimacy of the derivation*

In deriving the equation of motion (4.12) use was made of Fig. 6, which employs several tacit assumptions. One of these assumptions is that the two characteristics emanating from A strike the shock at P^- and Q^-. Additionally, it is supposed that $I(P^+)$ and $V(P^+)$ are known functions of x and t obtained from (4.1) and (4.2), (i.e., obtained from the state ahead of the shock). This amounts to assuming that the characteristics of the type C in Fig. 6 leave the t axis and also strike the shock. Both of these assumptions are valid if the shock speed $\dot{\xi}$ lies between the wave speeds ahead and behind the shock, i.e., if

$$\frac{1}{\sqrt{LC}} < \dot{\xi} < \frac{1}{\sqrt{L_1C}}. \qquad (4.13)$$

It will now be shown that this is the case, at least in the beginning of the development of the shock, by demonstrating that $\dot{\xi}(P)$ obeys (4.13). If the point P has the coordinates $(\xi(T), T)$, then, using (4.1) and (4.2), along with

$$\phi(t) = \phi(0) + \phi'(0)t,$$

$$\text{with} \qquad \phi(0) = \left(R + \sqrt{\frac{L}{C}}\right)I_c, \qquad (4.14)$$

(where we suppose that the initial point of the shock corresponds to $t = 0$ in (4.12)) and letting $t \to 0$ yields, after some manipulation, the following cubic in $\dot{\xi}(0)$,

$$\begin{aligned}0 = \dot{\xi}(0)[(\sqrt{L} + \sqrt{L_1})C\,\sqrt{L_1}\,\dot{\xi}^2(0) \\ - \sqrt{C}\,(\sqrt{L_1} - \sqrt{L})\dot{\xi}(0) - 2]. \qquad (4.15)\end{aligned}$$

The positive root of (4.15) is found to be

$$\dot{\xi}(0) = \frac{1}{\sqrt{C}\,(\frac{1}{2}(\sqrt{L} + \sqrt{L_1}))}. \qquad (4.16)$$

This root is seen to lie between $1/\sqrt{LC}$ and $1/\sqrt{L_1C}$. The equations (4.15) and (4.16) exhibit the interesting fact that the initial slope of the shock is independent of $\phi(t)$. In fact, the initial shock speed is apparently a curious kind of average of the speeds that are possible.

- *Straight-line shock*

A straight-line shock ($\ddot{\xi} \equiv 0$) occurs in at least two ways: (a) $\phi(t)$ is piecewise constant, (b) $\phi(t)$ is linear. These two cases will now be considered.

(*a*) *Piecewise constant* $\phi(t)$

Consider the case where $\phi(t)$ is a step function.

$$\phi(t) = \begin{cases}\theta_0 I_c < I_c(R + \sqrt{L/C}), & t < 0, \\ \theta_1 I_c > I_c(R + \sqrt{L/C}), & t > 0,\end{cases}$$

where $t = 0$ is the time when the shock first appears. (The inequalities here guarantee that the applied voltage is less than critical for $t < 0$ and greater than critical for $t > 0$.) Now the shock equation (4.12) becomes a quadratic in $\dot{\xi}$:

$$\begin{aligned}0 = {} & \dot{\xi}^2\left[-\theta_0 C\sqrt{LL_1} + \theta_1 CL_1 + (L - L_1)\sqrt{L_1C}\right] \\ & + \dot{\xi}(L - L_1)\left(1 - \frac{\theta_0\sqrt{C}}{\sqrt{L_1} + \sqrt{L}}\right) + \theta_0 - \theta_1. \qquad (4.17)\end{aligned}$$

Thus ξ is a constant equal to the positive root of this quadratic, and the shock is a straight line. The argument concerning inequality (4.13) is now no longer valid since ϕ was assumed to be differentiable for that argument. However, one may show that the positive root of (4.17) obeys inequality (4.13) as well.

In this situation of a straight-line shock the current and voltage in the region behind the shock are easily computed. Using Fig. 6, but allowing A to be anywhere in the region behind the shock, the equations (4.4) to (4.9) are solved for $I(A)$ and $V(A)$. The observations that V and I ahead of the shock (as given in (4.1) and (4.2)) are now constants (because ϕ is piecewise constant), and that $\alpha(\xi)$, $\beta(\xi)$, and $\gamma(\xi)$, as given in (3.12), are also constants, enable the following simple expression for $I(A)$ to be obtained:

$$I(A) = \left[\theta_0 \frac{1 - CL\xi^2}{R + \sqrt{\frac{L}{C}}} + C\xi^2(L - L_1)\right] \frac{I_c}{1 - CL_1\xi^2}. \tag{4.18}$$

From this and (4.13) one can easily show that $I(A) > I_c$. In a similar manner we obtain

$$V(A) = \left[\theta_0 \frac{\sqrt{\frac{L}{C}}(1 - CL_1\xi^2) - \xi(L - L_1)}{R + \sqrt{\frac{L}{C}}} + \xi(L - L_1)\right] \frac{I_c}{1 - CL_1\xi^2}. \tag{4.19}$$

Notice that while ahead of the shock the waves move only to the right, they move in both directions behind the shock. Thus, behind the shock we have reflected voltage and current waves, V_r and I_r, given by

$$V_r = \frac{V - \sqrt{L_1/C}\, I}{2} \tag{4.20}$$

$$I_r = -\sqrt{C/L_1}\, V_r. \tag{4.21}$$

The situation when ϕ is reduced from its larger value $\theta_1 I_c$ back to its lesser value $\theta_0 I_c$ will now be considered. Figure 7 anticipates the resulting situation. (Recall that $R = \sqrt{L_1/C}$.)

At the point on the t axis where ϕ is decreased, there emanates a fast wave, but above this fast wave (speed $1/\sqrt{L_1C}$) there are only slow waves (speed $1/\sqrt{LC}$). If this is the case, the line must unsaturate with the speed of the fast wave, leaving a line with specific inductance L. The current I above the last fast wave AB must be determined, and it must be verified that its value is not greater than I_c. It will be seen that the region ABC in Fig. 7 is actually somewhat indeterminate in the sense that in this region $I \equiv I_c$.

To determine I in the region above BA, consider two cases: Case (i), the point in question lies below BC; Case (ii), above BC. The method of computation of I in these two cases is illustrated in Fig. 8, where a portion of Fig. 7 is reproduced. Using the shock relations and characteristic equations pertinent to these diagrams we obtain for cases (i) and (ii), respectively:

Case (i)

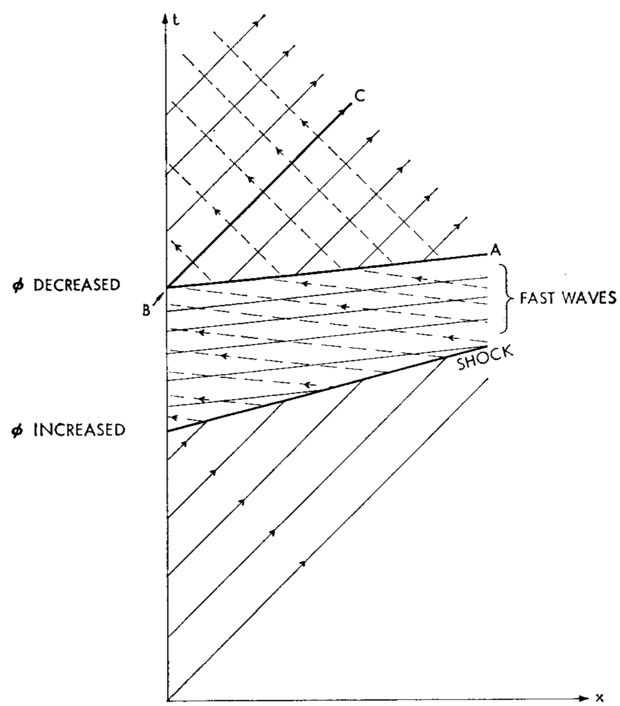
$$I(P) = I_c \tag{4.22}$$

$$V(P) = \left[\theta_0 \frac{\sqrt{L/C}(1 - CL_1\xi^2) - \xi(L - L_1)}{R + \sqrt{L/C}} + \xi(L - L_1)\right] \frac{I_c}{1 - CL_1\xi^2}. \tag{4.23}$$

Case (ii)

$$I(P) = I_c\Bigg[\theta_0 + \sqrt{L/C} - \frac{1}{1 - CL_1\xi^2} \cdot\left(\theta_0 \frac{\sqrt{L/C}(1 - CL_1\xi^2) + \xi(L - L_1)}{R + \sqrt{L/C}} + \xi(L - L_1)\right)\Bigg]\Big/\left[R + \sqrt{L/C}\right] \tag{4.24}$$

Figure 7 **Configuration showing relation between fast and slow waves** (*cf. Figure 5*).



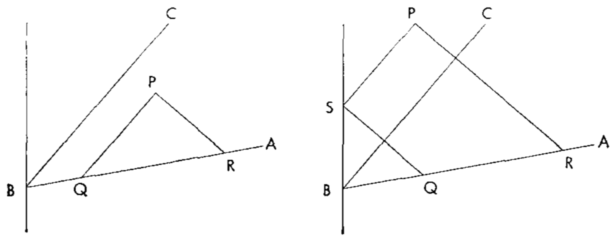


Figure 8 **Configuration of points utilized in the computation of current behind shock wave.**

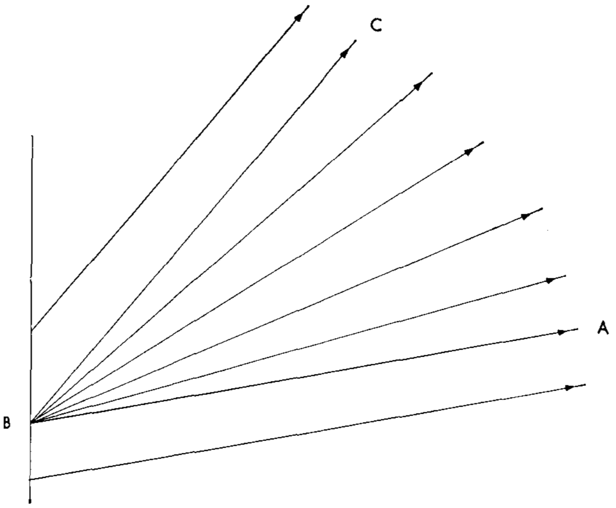


Figure 9 **Rarification wave behind shock.**

$$V(P) = I_c\left[\theta_0 \frac{\sqrt{L/C}\,(1 - CL_1\xi^2) + \xi(L - L_1)}{R + \sqrt{L/C}} + \xi(L - L_1)\right] \Big/ (1 - CL_1\xi^2). \quad (4.25)$$

A computation shows that the $I(P)$ as given in (4.24) is not greater than I_c.

The existence of the region ABC in Fig. 7 where $I \equiv I_c$ is due to the jump discontinuity in inductance from L_1 to L. If in the vicinity of $I = I_c$, L changed smoothly from L to L_1, then the region ABC would exhibit a gradual desaturation and be covered by a fan of characteristics of speeds varying from $1/\sqrt{LC}$ to $1/\sqrt{L_1C}$ as accompanies a typical rarifaction wave (see Fig. 9).

This situation yields a slower desaturation rate. Thus, while the analysis here shows a desaturation which moves with the speed $1/\sqrt{L_1C}$, since $I \equiv I_c$ in the region ABC, the line does not desaturate at this rate. It certainly desaturates with the slower speed $1/\sqrt{LC}$. However, a more careful approximation to the inductance $L = L(I)$ would give an intermediate rate.

(b) Linear $\phi(t)$

A straight-line shock occurs also when $\phi(t)$ is linear. If a linear $\phi(t)$ is inserted into the equation of shock motion (4.12), where $t = 0$ is the time at which $\phi(t)$ becomes critical, the following nonlinear ordinary differential equation for ξ is found to prevail:

$$\sqrt{CL_1}\,\xi^2\,[t - \sqrt{C}\,(\sqrt{L_1} - \sqrt{L})\xi] + \xi[t - \sqrt{CL}\,\xi] + \xi = 0. \quad (4.26)$$

Notice that the solution $\xi(t)$ of this equation is independent of the slope of $\phi(t)$. The solution of this differential equation is

$$\xi(t) = \frac{2}{\sqrt{C}\,(\sqrt{L} + \sqrt{L_1})}\,t. \quad (4.27)$$

This is the same solution as (4.16).

Appendix. Large array of films

In this Appendix a transmission line loaded with a periodically spaced set of films, as in Fig. 10, is considered.

By making certain approximations it will be shown that waves of long length in this line (compared to the length s of a strip) obey a set of transmission line equations, exactly as the case of a single film, with an appropriately adjusted value of specific inductance and specific capacity. The approximations make tacit use of the assumption that the TEM* character of the wave propagation is essentially preserved throughout the process. Thus, the analysis of the periodic array described here is absorbed into the analysis of the model of the single film, which is carried out in Section 4. The results are that waves that are long compared to s and d obey a set of transmission line equations with specific capacity and inductance given respectively by

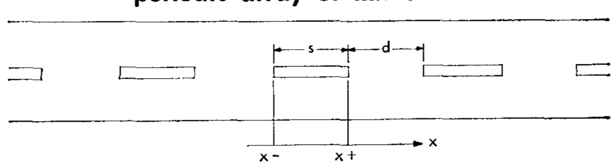
$$C_E = \left(1 + \frac{s}{d}\right)C$$

$$L_E = L + \frac{s}{d}\,L(I).$$

L and C are the specific inductance and capacity of the line in a section where the film is not present, and $L(I)$ is the specific inductance of the line in a section where the film is present.

* Transverse electromagnetic.

Figure 10 **Configuration of a line loaded with a periodic array of films.**



As $s \rightarrow 0$ in these relations, C_E and L_E tend to C and L of a transmission line which does not contain magnetic strips. However, as $d \rightarrow 0$ C_E and L_E do not tend to C and $L(I)$. Both of these occurrences are to be expected since, as a starting point for the derivation, the transmission line equations (A.1) and (A.2) were used in regions without films, and the four-pole equations (A.7) and (A.8) in regions with films. The four-pole equations are recoverable (in a sense) as $d \rightarrow 0$. The nonlinear transmission line equations, in regions with films, are not recoverable from the four-pole equations. A fortiori these transmission line equations are not recoverable from the derived equivalent line equations.

The details of the derivation now follow.

In sections of the line where the film is not present, the transmission line equations are

$$I_x = -C V_t \tag{A.1}$$

$$V_x = -L I_t. \tag{A.2}$$

In the remaining parts of the line the equations (2.20), written as

$$I_x = -C V_t \tag{A.3}$$

$$V_x = -L(I) I_t \tag{A.4}$$

will prevail.

Let x^- and x^+ denote the coordinates of the end points of a typical film, and let

$$[Y] = Y^+ - Y^- = Y(x^+, t) - Y(x^-, t) \tag{A.5}$$

$$\bar{Y} = \frac{Y^+ + Y^-}{2}, \tag{A.6}$$

where $Y(x, t)$ is any function of x and t. Now integrating (A.3) and (A.4) over the strip yields approximately

$$[I] = -sC \bar{V}_t \tag{A.7}$$

$$[V] = -sL(\bar{I}) \bar{I}_t. \tag{A.8}$$

The strip has therefore been replaced by a four-pole since the current and voltage on one end is directly related to the current and voltage on the other end by (A.7) and (A.8).

Equations (A.3) and (A.4) may be combined by differentiation into

$$I_{xx} = C(L(\bar{I}) \bar{I}_t)_t \tag{A.9}$$

which, by integration over the strip, gives approximately

$$[I_x] = sC(L(\bar{I}) \bar{I}_t)_t. \tag{A.10}$$

Now Eq. (A.2) gives

$$[V_x] = -L[I_t] = -L \int I_{tx}\, dx, \tag{A.11}$$

where again integration is over a strip. Computing I_{tx}

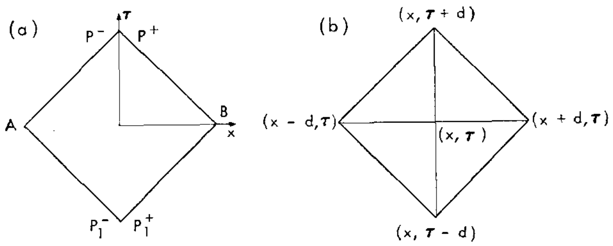


Figure 11 **(a) Rhombus of characteristics of wave equation. (b) Coordinates for rhombus of characteristics.**

from (A.3) enables (A.11) to be written in the form

$$[V_x] = CLs \bar{V}_{tt}. \tag{A.12}$$

Introducing a new time coordinate

$$\sqrt{CL}\, \tau = t, \tag{A.13}$$

Eqs. (A.1), (A.2), (A.7), (A.8), (A.10), and (A.12) may be rewritten as

$$\left.\begin{aligned} I_{xx} &= I_{\tau\tau} \\ V_{xx} &= V_{\tau\tau} \end{aligned}\right\} \text{outside of film regions and} \tag{A.14}$$

$$\left.\begin{aligned} [I_x] &= s\left(\frac{L(\bar{I})}{L} \bar{I}_\tau\right)_\tau \\ [V_x] &= s \bar{V}_{\tau\tau} \\ [I] &= -s\sqrt{\frac{C}{L}}\, \bar{V}_\tau \\ [V] &= -s\sqrt{\frac{L}{C}}\, \frac{L(\bar{I})}{L}\, \bar{I}_\tau \end{aligned}\right\} \text{across film strips.} \tag{A.15}$$

The strips are considered to be collapsed, each to a single point, and the equations (A.15) which relate V and I and their derivatives at the opposite ends of a strip have replaced the transmission line equations (A.3) and (A.4).

Now Gauss' theorem for a solution of the wave equation,

$$0 = \iint (V_{xx} - V_{\tau\tau})\, dx\, d\tau$$

$$= \oint (V_x \cos(n, x) - V_t \cos(n, t))\, ds, \tag{A.16}$$

is applied to V and I in the domains AP_1P and PBP_1 of Fig. 11.

In this Figure P and P_1 lie on the line $x =$ constant (to which a strip has been collapsed). The remaining lines have slope ± 1 (i.e., are portions of characteristics of the wave equation). This application of Gauss' theorem gives

$$0 = -\int_{P_1}^{P} I_x^- \, d\tau + I(P^-) - 2I(A) + I(P_1^-) \quad \text{(A.17)}$$

and

$$0 = \int_{P_1}^{P} I_x^+ \, d\tau + I(P^+) - 2I(B) + I(P_1^+). \quad \text{(A.18)}$$

Adding these equations gives

$$0 = \int_{P_1}^{P} [I_x] \, d\tau + 2\bar{I}(P) - 2I(A) - 2I(B) + 2\bar{I}(P_1). \quad \text{(A.19)}$$

Inserting the value of $[I_x]$, as given in (A.15), into this equation yields, upon integration,

$$0 = \frac{s}{2} \frac{L(\bar{I})}{L} \bar{I}_\tau \Big|_{P_1}^{P} + \bar{I}(P) - I(A) - I(B) + \bar{I}(P_1). \quad \text{(A.20)}$$

Here $F(\bar{I})_{P_1}^{P} \cdot \equiv \cdot F(\bar{I}(P)) - F(\bar{I}(P_1))$.
In a similar manner

$$0 = \frac{s}{2} \bar{V}_\tau \Big|_{P_1}^{P} + \bar{V}(P) - V(A) - V(B) + \bar{V}(P_1) \quad \text{(A.21)}$$

is obtained.

Now consider Fig. 11b, which is the same as Fig. 10 except that the lengths of the line segments are fixed so that A and B lie on strips.

For these domains (A.20) becomes

$$0 = \frac{s}{2}\left[\frac{L(\bar{I})}{L} \bar{I}_\tau(x, \tau + d) - \frac{L(I)}{L} \bar{I}_\tau(x, \tau - d)\right] + \bar{I}(x, \tau + d) + \bar{I}(x, \tau - d) - I(x - d, \tau) - I(x + d, \tau). \quad \text{(A.22)}$$

Now

$$I(x - d, \tau) + I(x + d, \tau) = \bar{I}(x - d, \tau) + \bar{I}(x + d, \tau) + \frac{I((x - d)^+, \tau) - I((x - d)^-, \tau)}{2} + \frac{I((x + d)^-, \tau) - I((x + d)^+, \tau)}{2}. \quad \text{(A.23)}$$

If the third equation in (A.15) is inserted into (A.23), it becomes

$$I(x - d, \tau) + I(x + d, \tau) = \bar{I}(x - d, \tau) + \bar{I}(x + d, \tau) - \frac{s}{2}\sqrt{\frac{C}{L}}\,[\bar{V}_\tau(x - d, \tau) - \bar{V}_\tau(x + d, \tau)]. \quad \text{(A.24)}$$

Inserting this into (A.22) and dividing the resulting equation by d^2 gives

$$0 = \frac{s}{d}\left[\frac{L(\bar{I})\bar{I}_\tau(x, \tau + d) - L(\bar{I})\bar{I}_\tau(x, \tau - d)}{2\,dL}\right] + \frac{\bar{I}(x, \tau + d) + \bar{I}(x, \tau - d)}{d^2} - \frac{\bar{I}(x + d, \tau) + \bar{I}(x - d, \tau)}{d^2} - \frac{s}{d}\sqrt{\frac{C}{L}}\left[\frac{\bar{V}_\tau(x + d, \tau) - \bar{V}_\tau(x - d, \tau)}{2d}\right]. \quad \text{(A.25)}$$

Now let

$$\theta = \lim_{s, d \to 0} \frac{s}{d}. \quad \text{(A.26)}$$

Then taking the limit as $s, d \to 0$, (5.25) becomes

$$0 = \theta\left[\frac{L(\bar{I})}{L} \bar{I}_\tau\right]_\tau + \bar{I}_{\tau\tau} - \bar{I}_{xx} - \theta\sqrt{\frac{C}{L}}\,\bar{V}_{\tau x}. \quad \text{(A.27)}$$

In a similar manner the equation

$$0 = (1 + \theta)\bar{V}_{\tau\tau} - \bar{V}_{xx} - \theta\sqrt{\frac{L}{C}}\left(\frac{L(\bar{I})}{L} \bar{I}_\tau\right)_x \quad \text{(A.28)}$$

is obtained using (5.21) as a starting point.

The following two first-order differential equations

$$\bar{I}_x + (1 + \theta)C\,\bar{V}_t = 0 \quad \text{(A.29)}$$

$$\bar{V}_x + (L + \theta L(I))\bar{I}_t = 0 \quad \text{(A.30)}$$

give rise, by differentiation, to the original second-order equations (A.27) and (A.28).

Now the class of solutions of (A.27) and (A.28) contains the solution of the transmission line equations (A.29) and (A.30). However, if (A.29) and (A.30) are used instead of (A.27) and (A.28), at least some of the solutions of the latter will be characterized. Thus, a periodic array of strips in a line behaves (for wave lengths long compared to s and d) like an equivalent line loaded with a single long strip where the effective specific capacity and specific inductance of the equivalent line are

$$C_E = (1 + \theta)C = \left(1 + \frac{s}{d}\right)C \quad \text{(A.31)}$$

$$L_E = L + \theta L(\bar{I}) = L + \frac{s}{d} L(\bar{I}). \quad \text{(A.32)}$$

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