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Comments on the Presence Function of Gazalé†

This Letter is an extension of the paper published by Gazalé[1] in this Journal in April 1957.

Let ϕ be a (Boolean) function of the (Boolean) variables $x_1, \cdots, x_n$. A finite system $F_1, \cdots, F_M$ of functions of the same variables will be termed a representation of ϕ if

$$\phi = F_1 + \cdots + F_M , \tag{1}$$

identically in the variables $x_1, \cdots, x_n$. A representation (1) will be termed *T-irredundant* (term-wise irredundant) if no proper subsystem of $F_1, \cdots, F_M$ is a representation of ϕ. In a variety of important situations, one has an initial representation $F_1, \cdots, F_M$ which is *not* known to be T-irredundant; and there arises the problem to find all those subsystems of $F_1, \cdots, F_M$ which *do* constitute T-irredundant representations for ϕ. Gazalé proposed an interesting approach (based on the use of a certain auxiliary function which he called *the presence function*) for the special case when $F_1, \cdots, F_M$ are products of x-literals. Now in some programs (related to the simplification of multiple-output networks) one has to deal with the more general case when $F_1, \cdots, F_M$ are *not* products of literals but quite general functions of $x_1, \cdots, x_n$. Also, for purposes of actual programming it was found necessary to set up an explicit formula for the presence function. The purpose of this note is to present (and to prove) the results obtained by the writer after having studied the significant paper of Gazalé.

Definitions

We assume that an initial representation (see (1)) is given for ϕ. We first introduce an auxiliary Boolean function Δ of the auxiliary Boolean variables $v_1, \cdots, v_M$ by the formula

$$\Delta(v_1, \cdots, v_M) = \begin{cases} 1 & \text{if } v_1 = \cdots = v_M , \\ 0 & \text{otherwise} . \end{cases} \tag{2}$$

This function is a generalization of the well-known *Kronecker delta*. Next, we put $K = 2^n$ (where n is the number of the x-variables), and enumerate (in any desired manner) the K possible substitutions (of binary digits) for the variables $x_1, \cdots, x_n$ as $\xi_1, \cdots, \xi_K$. If ψ is any (Boolean) function of $x_1, \cdots, x_n$, we shall denote by $\psi(\xi_k)$ the value of ψ for the substitution ξ_k. Next we introduce M auxiliary variables $\sigma_1, \cdots, \sigma_M$ and put (for $1 \leqq k \leqq K$)

$$B_k = \Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] + \sum_{m=1}^{M} F_M(\xi_k)\sigma_m . \tag{3}$$

$$S(\sigma_1, \cdots, \sigma_M) = \prod_{k=1}^{K} B_k . \tag{4}$$

The theorem to be proved below will show that S corresponds to the *presence function* of Gazalé, extended to the general case considered here. Let us note that (2), (3), (4) yield an *explicit formula* for S; actual use (as a subroutine of larger programs) showed that this formula furnishes a fast program, adequate even for computers in the medium speed range.

Lemma 1

Let $m_1, \cdots, m_R$ be subscripts such that $1 \leqq m_1 < \cdots < m_R \leqq M$ and

$$\phi = F_{m_1} + \cdots + F_{m_R} . \tag{5}$$

Then the product

$$\sigma_{m_1} \cdots \sigma_{m_R} \tag{6}$$

is an implicant of $S(\sigma_1, \cdots, \sigma_M)$.

- *Proof*

Consider any substitution σ^0 of (binary digits) $\sigma_1{}^0, \cdots, \sigma_M{}^0$ for $\sigma_1, \cdots, \sigma_M$ such that

$$\sigma_{m_1}{}^0 \cdots \sigma_{m_R}{}^0 = 1 . \tag{7}$$

We have to show that $S = 1$ for this substitution. Consider any one of the expressions B_k (see (3)), and denote by $B_k{}^0$ the expression obtained by replacing $\sigma_1, \cdots, \sigma_M$ by $\sigma_1{}^0, \cdots, \sigma_M{}^0$.

- *Case 1*

$\Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] = 1$. Then clearly $B_k{}^0 = 1$.

- *Case 2*

$\Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] = 0$. Then $F_1(\xi_k), \cdots, F_M(\xi_k)$ do *not* have the same value (see (2)). Hence there exists a subscript m such that $F_m(\xi_k) = 1$. We have then, by (1), $\phi(\xi_k) = 1$; hence, by (5), we have an r

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such that $F_{m_r}(\xi_k) = 1$. But then the summation in (3) contains the term

$F_{m_r}(\xi_k)\sigma_{m_r}$,

which has the value 1 by (7). Thus $B_k{}^0 = 1$ in Case 2 also.

Thus we see that $B_k{}^0 = 1$ for $1 \leqq k \leqq K$; hence, by (4), $S = 1$, and the lemma is proved.

Lemma 2

Let $m_1, \cdots, m_R$ be subscripts such that $1 \leqq m_1 < \cdots < m_R \leqq M$, and $\sigma_{m_1}, \cdots, \sigma_{m_R}$ is an implicant of S. Then

$$\phi = F_{m_1} + \cdots + F_{m_R} . \quad (8)$$

• *Proof*

Make, for the σ-variables, the substitution

$$\sigma_{m_1}{}^0 = \cdots = \sigma_{m_R}{}^0 = 1, \;\; \sigma_m{}^0 = 0 \text{ for } m \neq m_1, \cdots, m_R . \quad (9)$$

Then $\sigma_{m_1}{}^0 \cdots \sigma_{m_R}{}^0 = 1$; since, by assumption, $\sigma_{m_1} \cdots \sigma_{m_R}$ is an implicant of S, it follows that we have now $S = 1$. Using the notation $B_k{}^0$ as in the proof of Lemma 1, we have therefore (see (4))

$$B_k{}^0 = 1 \text{ for } 1 \leqq k \leqq K . \quad (10)$$

Now, in view of (9),

$$B_k{}^0 = \Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] + \sum_{r=1}^{R} F_{m_r}(\xi_k) . \quad (11)$$

Fix now k.

• *Case 1*

$\Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] = 1$. Then $F_1(\xi_k), \cdots, F_M(\xi_k)$ have the *same* value (0 or 1). By (1) we have then $\phi(\xi_k) = d$, and the summation in (11) also has the value d. Hence

$$\phi(\xi_k) = \sum_{r=1}^{R} F_{m_r}(\xi_k) . \quad (12)$$

• *Case 2*

$\Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] = 0$. Then $F_1(\xi_k), \cdots, F_M(\xi_k)$ do *not* have the same value. Hence there is a subscript m such that $F_m(\xi_k) = 1$. Hence, by (1), we have $\phi(\xi_k) = 1$. But, since $B_k{}^0 = 1$ by (10) and $\Delta[F_1(\xi_k), \cdots, F_M(\xi_k)] = 0$ (by assumption), it follows that the summation in (11) has the value 1 also. Thus (12) holds in Case 2 also.

Now since k was arbitrary, it follows that (12) holds for $1 \leqq k \leqq K$; that is, (8) holds for *every* substitution for $x_1, \cdots, x_n$, and Lemma 2 is proved.

Theorem

Let $m_1, \cdots, m_R$ be subscripts such that $1 \leqq m_1 \leqq \cdots < m_R \leqq M$. Then the functions $F_{m_1}, \cdots, F_{m_R}$ yield a T-irredundant representation for ϕ if and only if $\sigma_{m_1} \cdots \sigma_{m_R}$ is a prime implicant of S.

• *Proof*

Assume first that

$$\phi = F_{m_1} + \cdots + F_{m_R} \quad (13)$$

is a T-irredundant representation for ϕ. By Lemma 1, $\sigma_{m_1}, \cdots, \sigma_{m_R}$ is an implicant of S. Now, if $\sigma_{m_1} \cdots \sigma_{m_R}$ were *not* a prime implicant of S, then we would have a *subset* $\mu_1, \cdots, \mu_t$ of the indices $m_1, \cdots, m_R$ such that $t < R$ and $\sigma_{\mu_1} \cdots \sigma_{\mu_t}$ is an implicant of S. By Lemma 2, we would have then

$$\phi = F_{\mu_1} + \cdots + F_{\mu_t} ,$$

contradicting the assumption that $F_{m_1}, \cdots, F_{m_R}$ constitute a T-irredundant representation for ϕ.

Assume next that $\sigma_{m_1} \cdots \sigma_{m_R}$ is a *prime* implicant of S. By Lemma 2, the functions $F_{m_1}, \cdots, F_{m_R}$ constitute then a representation for ϕ. If this representation were not T-irredundant, then we would have a subset $\mu_1, \cdots, \mu_t$ of the indices $m_1, \cdots, m_R$ such that $t < R$ and $F_{\mu_1}, \cdots, F_{\mu_t}$ constitute a representation for ϕ. But then (by Lemma 1) $\sigma_{\mu_1} \cdots \sigma_{\mu_t}$ would be an implicant of S, in contradiction with the assumption that $\sigma_{m_1} \cdots \sigma_{m_R}$ is a *prime* implicant of S. The proof of the theorem is now complete.

The prime implicants of S

Since, in computing S, the terms B_k in which $\Delta = 1$ have the value 1 and can therefore be omitted, S appears as a *product of sums* of unbarred σ-variables. By a well-known theorem of Nelson, one finds then the prime implicants of S as follows. After the multiplication is carried out, one drops every σ-product which is a multiple of some other σ-product present; the surviving σ-products are then precisely the prime implicants of S.

Reference

1. M. J. Gazalé, *IBM Journal* 1, 171 (1957).

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