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Comment on "A Network Minimization Problem"

This communication is a comment on the Letter by F. P. Palermo on network minimization, published recently in the *IBM Journal.*[1]

With the use of complex numbers one can simplify the proofs considerably. Let $z = x + iy$ represent the point (x, y). We have to prove that if z_k, $k = 1, \cdots, N$ are non-collinear points, and m_k are positive numbers, then

$$F(z) = \sum_{1}^{N} m_k|z - z_k|$$

has one and only one minimum. The existence of a minimum is easily seen from a theorem of Weierstrass, since $F(z)$ is continuous in any closed circle and tends to infinity with $|z|$. From the triangle inequality follows for any α that $|z + \alpha - z_k| + |z - \alpha - z_k| \leqq 2|z - z_k|$ and hence $F(z + \alpha) + F(z - \alpha) \leqq 2F(z)$. We have strict inequality for $\alpha \neq 0$, since the points are not collinear. Thus $F(z)$ is strictly convex on any straight line and the uniqueness follows.

Even if the z_k are collinear, $F(z)$ has in general a unique minimum. Let $z_k = x_k$ be real, with $x_1 < x_2 < \cdots < x_N$. Then

$$F'(x) = \sum_{k \leqq j} m_k - \sum_{k \geqq j+1} m_k \equiv S_j$$

for $x_j < x < x_{j+1}$. If and only if $S_j = 0$ for some j, $F(x)$ has its least value on the interval $x_j \leqq x \leqq x_{j+1}$. When $S_j \neq 0$ for all values of j, let κ be the smallest index j for which $S_j > 0$. Then x_κ is the unique minimum point.

The proof of uniqueness can be extended. Let z_k and z be elements of a linear space with a distance $|z_1 - z_2|$, which obeys a strong triangle inequality:

$$|z_1| + |z_2| \geqq |z_1 + z_2| \qquad (1)$$

with equality (if and) only if $z_1 = kz_2$ with a positive k or if one element is zero.

Then $F(z + tz^*)$ is strictly convex as a function of the real parameter t for any elements z and z^*, if the points z_k are not collinear. Also for the case of collinearity an extension is possible.

The existence of a unique minimum thus holds, not only in any finite-dimensional space with the ordinary distance, but also in many functional spaces. It can be observed that (1) is sufficient, but not necessary for uniqueness. Let the largest of the absolute values of $x - x_k$ and $y - y_k$ be the distance $|z - z_k|$ between the complex numbers z and z_k. Then (1) fails and in general $F(z)$ has not a unique minimum, but if z_k are real, the uniqueness follows as above, if $S_j \neq 0$ for all values of j.

Reference

1. F. P. Palermo, "A Network Minimization Problem," *IBM Journal* **5**, 335 (October 1961).

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