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Solving a Matrix Game by Linear Programming

Abstract: This paper presents (1) a new characterization, via linear programming, of extreme optimal strategies of a matrix game and (2) a simple direct procedure for computing them. The first pertains to the neat formulas of L. S. Shapley and R. N. Snow for a "basic solution", and the second to the highly effective "simplex method" of G. B. Dantzig. Both are related to the author's "combinational equivalence" of matrices, the first through an optimal block-pivot transformation and the second through a suitably chosen succession of elementary pivot steps.

Introduction

Let

$$\begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{m1} & \cdots & g_{mn} \end{bmatrix}$$

be the payoff matrix of a two-person zero-sum game in normalized form. Let probabilities $p_1, \cdots, p_m$ (each $\geqq 0$, sum $= 1$) and $q_1, \cdots, q_n$ (each $\geqq 0$, sum $= 1$) constitute mixed strategies P and Q for the "payee" player I and the "payer" player II, respectively. Then the schema (schematic representation)

$$\begin{array}{c|ccc|c} & q_1 & \cdots & q_n & \\ \hline p_1 & g_{11} & \cdots & g_{1n} & = f_1 \\ \vdots & \vdots & & \vdots & \vdots \\ p_m & g_{m1} & \cdots & g_{mn} & = f_m \\ \hline & = e_1 & \cdots & = e_n & \end{array}$$

or

$$\begin{array}{c|c|c} & Q & \\ \hline P & G & = F \\ \hline & = E & \end{array}$$

exhibits player I's expected *gains*

$$\begin{matrix} p_1 g_{11} + \cdots + p_m g_{m1} = e_1 \\ \vdots \\ p_1 g_{1n} + \cdots + p_m g_{mn} = e_n \end{matrix} \quad \text{or} \quad PG = E$$

against II's columns, and player II's expected *losses*

$$\begin{matrix} g_{11} q_1 + \cdots + g_{1n} q_n = f_1 \\ \vdots \\ g_{m1} q_1 + \cdots + g_{mn} q_n = f_m \end{matrix} \quad \text{or} \quad GQ = F$$

against I's rows. In the above G-schema, inner (scalar) products of P with the columns of G produce the column equation-system $PG = E$, and inner products of the rows of G with Q produce the row equation-system $GQ = F$.

Let u denote the minimum among $e_1, \cdots, e_n$ and v the maximum among $f_1, \cdots, f_m$. Player I's objective is to choose his mixed strategy P so as to maximize the "floor" u under his expected gains $e_1, \cdots, e_n$. Player II's objective is to choose his mixed strategy Q so as to minimize the "ceiling" v over his expected losses $f_1, \cdots, f_m$. Necessarily $u \leqq v$. By the Minimax Theorem (1928) of John von Neumann (the Main Theorem in [1]), there must exist mixed strategies P and Q for which the "floor" u coincides with the "ceiling" v. This state of equilibrium is *optimal* for both players, assuming rational behavior. Such mixed strategies P and Q are called *optimal*; the equilibrium value $u = v$ is the (unique) *value* of the game.

Transferring attention from $e_1, \cdots, e_n$ and $f_1, \cdots, f_m$ to $e_1 - u = s_1(\geqq 0), \cdots e_n - u = s_n(\geqq 0)$ and $v - f_1 = t_1(\geqq 0), \cdots, v - f_m = t_m(\geqq 0)$, form the schema

$$
\begin{array}{c|c:ccc|l}
 & -v & q_1 & \cdots & q_n & \\
\hline
u & 0 & -1 & \cdots & -1 & = -1 \\
\hdashline
p_1 & 1 & g_{11} & \cdots & g_{1n} & = -t_1 \\
\vdots & \vdots & \vdots & & \vdots & \quad \vdots \\
p_m & 1 & g_{m1} & \cdots & g_{mn} & = -t_m \\
\hline
 & =1 & = s_1 & \cdots & = s_n &
\end{array}
$$

which describes column and row equation-systems as follows:

$$
\begin{array}{rcl}
p_1 + \cdots + p_m & = & 1 \\
-u + p_1 g_{11} + \cdots + p_m g_{m1} & = & s_1 \\
\vdots & & \vdots \\
-u + p_1 g_{1n} + \cdots + p_m g_{mn} & = & s_n
\end{array}
$$

and

$$
\begin{array}{rcl}
-q_1 - \cdots - q_n & = & -1 \\
-v + g_{11} q_1 + \cdots + g_{1n} q_n & = & -t_1 \\
\vdots & & \vdots \\
-v + g_{m1} q_1 + \cdots + g_{mn} q_n & = & -t_m \, .
\end{array}
$$

This schema may be written more compactly as

$$
\begin{array}{c|c:c|l}
 & -v & Q & \\
\hline
u & 0 & -1 & = -1 \\
\hdashline
P & 1 & G & = -T \\
\hline
 & =1 & =S &
\end{array}
\quad \text{or} \quad
\begin{array}{c|c|l}
 & \bar{Q} & \\
\hline
\bar{P} & \bar{G} & = -\bar{T}, \\
\hline
 & = \bar{S} &
\end{array}
$$

where $\bar{G}$ denotes the matrix obtained by bordering G with a column of 1's, a row of -1's, and a corner entry 0, where $\bar{P}$ denotes the vector $[u, P]$, and so on. Player I's objective is to maximize u subject to the column equation-system $\bar{P}\bar{G} = \bar{S}$ and the inequalities $P \geqq 0$, $S \geqq 0$. Player II's objective is to minimize v subject to the row equation-system $\bar{G}\bar{Q} = -\bar{T}$ and the inequalities $Q \geqq 0$, $T \geqq 0$. (A vector inequality holds in each component—e.g., $P \geqq 0$ means that $p_1 \geqq 0, \cdots, p_m \geqq 0$.)

Note that the bordered matrix $\bar{G}$, just as the payoff matrix G, is replaced by its negative-transpose if the rôles of players I and II are interchanged.

Block-pivot transformation to dual linear programs

The problems of players I and II, as summarized in the $\bar{G}$-schema above, can be transformed into an equivalent pair of dual linear programs of the following form:

Maximize $u = -XB + d$ constrained by $XA \geqq C$, $X \geqq 0$.

Minimize $v = -CY + d$ constrained by $AY \leqq B$, $Y \geqq 0$.

To this end, take a square submatrix G_{11} of G such that its bordered counterpart $\bar{G}_{11}$ is a *non-singular* square submatrix of $\bar{G}$. (This is always possible; for example, det $\bar{G}_{11} = 1$ when G_{11} is any square submatrix of order one, i.e., just a single entry of G.) Let G_{12}, G_{21}, G_{22} and $\bar{G}_{12}$, $\bar{G}_{21}$, $\bar{G}_{22} = G_{22}$ be the submatrices of G and $\bar{G}$, respectively, that remain when G_{11} and $\bar{G}_{11}$ are removed. Then the $\bar{G}$-schema can be rearranged as follows (by suitable permutation of rows and/or columns):

$$
\begin{array}{c|c:c|c|l}
 & -v & Q_1 & Q_2 & \\
\hline
u & 0 & -1 & -1 & = -1 \\
\hdashline
P_1 & 1 & G_{11} & G_{12} & = -T_1 \\
\hline
P_2 & 1 & G_{21} & G_{22} & = -T_2 \\
\hline
 & =1 & =S_1 & =S_2 &
\end{array}
$$

or

$$
\begin{array}{c|c|c|l}
 & \bar{Q}_1 & Q_2 & \\
\hline
\bar{P}_1 & \bar{G}_{11} & \bar{G}_{12} & = -\bar{T}_1 \\
\hline
P_2 & \bar{G}_{21} & G_{22} & = -T_2 \\
\hline
 & =\bar{S}_1 & =S_2 &
\end{array}
\, .
$$

In the latter schema, solve

$$\bar{P}_1\bar{G}_{11} + P_2\bar{G}_{21} = \bar{S}_1 \quad \text{and} \quad \bar{G}_{11}\bar{Q}_1 + \bar{G}_{12}Q_2 = -\bar{T}_1$$

for $\bar{P}_1$ and $-\bar{Q}_1$ ($\bar{G}_{11}$ being nonsingular) to get

$$\bar{S}_1\bar{G}_{11}^{-1} - P_2\bar{G}_{21}\bar{G}_{11}^{-1} = \bar{P}_1$$

and

$$\bar{G}_{11}^{-1}\bar{T}_1 + \bar{G}_{11}^{-1}\bar{G}_{12}Q_2 = -\bar{Q}_1 \, .$$

Now, substitute for $\bar{P}_1$ and $\bar{Q}_1$ in

$$\bar{P}_1\bar{G}_{12} + P_2G_{22} = S_2 \quad \text{and} \quad \bar{G}_{21}\bar{Q}_1 + G_{22}Q_2 = -T_2$$

to get

$$\bar{S}_1\bar{G}_{11}^{-1}\bar{G}_{12} + P_2(G_{22} - \bar{G}_{21}\bar{G}_{11}^{-1}\bar{G}_{12}) = S_2$$

and

$$-\bar{G}_{21}\bar{G}_{11}^{-1}\bar{T}_1 + (G_{22} - \bar{G}_{21}\bar{G}_{11}^{-1}\bar{G}_{12})Q_2 = -T_2 \ .$$

These new equation-systems, equivalent to the old (i.e., having the same solutions), are described by the schema

$$\begin{array}{c|c:c|c|c} & 1 & T_1 & Q_2 & \\ \hline 1 & d & -C_1 & -C_2 & = v \\ \hdashline S_1 & -B_1 & A_{11} & A_{12} & = -Q_1 \\ \hline P_2 & -B_2 & A_{21} & A_{22} & = -T_2 \\ \hline & = u & = P_1 & = S_2 & \end{array}$$

or

$$\begin{array}{c|c|c|c} & \bar{T}_1 & Q_2 & \\ \hline \bar{S}_1 & \bar{A}_{11} & \bar{A}_{12} & = -\bar{Q}_1 \\ \hline P_2 & \bar{A}_{21} & A_{22} & = -T_2 \\ \hline & = \bar{P}_1 & = S_2 & \end{array} \ ,$$

where

$$\bar{A}_{11} = \bar{G}_{11}^{-1} \ , \qquad \bar{A}_{12} = \bar{G}_{11}^{-1}\bar{G}_{12} \ ,$$

$$\bar{A}_{21} = -\bar{G}_{21}\bar{G}_{11}^{-1} \ , \qquad A_{22} = G_{22} - \bar{G}_{21}\bar{G}_{11}^{-1}\bar{G}_{12} \ .$$

The nonsingular submatrix $\bar{G}_{11}$, which plays the central rôle in the transformation from the old schema to the new schema, is called a *block-pivot*. The matrix $\bar{G}$ of the old schema and the matrix $\bar{A}$ of the new schema are "combinatorially equivalent" (see [2], Theorem 4), a fundamental relation of broad applicability.

Let

$$X = [S_1, P_2], \qquad U = [P_1, S_2]$$

and

$$Y = \begin{bmatrix} T_1 \\ Q_2 \end{bmatrix}, \qquad V = \begin{bmatrix} Q_1 \\ T_2 \end{bmatrix}.$$

Then the new schema, rewritten as

$$\begin{array}{c|c:c|c} & 1 & Y & \\ \hline 1 & d & -C & = v \\ \hdashline X & -B & A & = -V \\ \hline & = u & = U & \end{array} \quad \text{or} \quad \begin{array}{c|c|c} & \bar{Y} & \\ \hline \bar{X} & \bar{A} & = -\bar{V}, \\ \hline & = \bar{U} & \end{array}$$

pertains to the dual linear programs:

Maximize $u = -XB + d$ constrained by

$$U = XA - C \geqq 0, X \geqq 0.$$

Minimize $v = -CY + d$ constrained by

$$-V = AY - B \leqq 0, Y \geqq 0.$$

Thus the problems of players I and II, as summarized in the $\bar{G}$-schema, have been transformed via a block-pivot $\bar{G}_{11}$ into a (combinatorially) equivalent pair of dual linear programs, as summarized in the $\bar{A}$-schema.

The term *feasible* is applied to solutions X, U, u and Y, V, v of the column and row equation-systems of the $\bar{A}$-schema if they satisfy the constraint inequalities $X \geqq 0$, $U \geqq 0$ and $Y \geqq 0$, $V \geqq 0$ of the respective linear programs. For any pair of feasible solutions, $u \leqq v$ necessarily, as may readily be verified; if $u = v$, then both feasible solutions are *optimal.*

Note that any mixed strategies P and Q give rise to feasible solutions

$$X_1 = S_1, X_2 = P_2, U_1 = P_1, U_2 = S_2,$$

$$u = \text{minimal component of } PG$$

and

$$Y_1 = T_1, Y_2 = Q_2, V_1 = Q_1, V_2 = T_2,$$

$$v = \text{maximal component of } GQ,$$

where $S = PG - u$ and $T = v - GQ$.

Optimal block-pivots and Shapley-Snow basic kernels

The "basic solutions" $X = 0$, $U = -C$, $u = d$ and $Y = 0$, $V = B$, $v = d$ of the column and row equation-systems of the $\bar{A}$-schema, above, are feasible if and only if $-C \geqq 0$ and $B \geqq 0$, respectively. When both these conditions hold simultaneously, the basic solutions are optimal, since $u = v = d$. Then

$$P_1 = -C_1 \ , P_2 = 0 \qquad \text{and} \qquad Q_1 = B_1 \ , Q_2 = 0$$

constitute optimal strategies for players I and II, and $u = v = d$ is the value of the game. Accordingly, when both $-C \geqq 0$ and $B \geqq 0$, let the term *optimal* be applied also to a block-pivot $\bar{G}_{11}$, through which the $\bar{G}$-schema is transformed into an $\bar{A}$-schema with "optimal basic solutions."

Let

$$\text{adj } G_{11} = ||h_{ij}||,$$

h_{ij} being the cofactor of the j,i-th entry of G_{11}.

Then, since

$$\begin{bmatrix} d & -C_1 \\ -B_1 & A_{11} \end{bmatrix} = \bar{A}_{11} = \bar{G}_{11}^{-1} = \text{adj } \bar{G}_{11}/\det \bar{G}_{11}$$

$$= \text{adj} \begin{bmatrix} 0 & -1 \\ 1 & G_{11} \end{bmatrix} \div \det \begin{bmatrix} 0 & -1 \\ 1 & G_{11} \end{bmatrix},$$

it can be shown that

$$d = \det G_{11}/\Sigma_i \, \Sigma_j \, h_{ij},$$

the j-th entry of $-C_1 = \Sigma_i \, h_{ij}/\Sigma_i \, \Sigma_j \, h_{ij}$,

the i-th entry of $B_1 = \Sigma_j \, h_{ij}/\Sigma_i \, \Sigma_j \, h_{ij}$.

For $\bar{G}_{11}$ an optimal block-pivot, these are the "basic solution" formulas of L. S. Shapley and R. N. Snow [3]: the square submatrix G_{11} of G is a *basic kernel* and the mixed strategies $P_1 = -C_1$, $P_2 = 0$, and $Q_1 = B_1$, $Q_2 = 0$ are *extreme* optimal strategies for players I and II. (Also see [4], p. 85, where the Shapley-Snow formulas are derived from "systems of equated constraints.") Each basic kernel G_{11} of G gives rise to a nonsingular $\bar{G}_{11}$ (since $\det \bar{G}_{11} = \Sigma_i \, \Sigma_j \, h_{ij} \neq 0$), which constitutes an optimal block-pivot as described above. Note that a basic kernel G_{11} is singular if and only if the game value d is zero, but that $\bar{G}_{11}$ is nonsingular without exception.

• *Example 1*

As payoff matrix, take

$$G = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where the entries in this two-by-two matrix are such that

$$\alpha \geqq \beta, \qquad \alpha \geqq \gamma, \qquad \delta \geqq \beta, \qquad \delta \geqq \gamma$$

and

$$\alpha - \beta - \gamma + \delta = 1.$$

Make the block-pivot transformation from the $\bar{G}$-schema, at left, to the $\bar{A}$-schema, at right. Here the block-pivot $\bar{G}_{11} = \bar{G}$ and

$$\bar{A} = \bar{A}_{11} = \bar{G}_{11}^{-1} = \text{adj } \bar{G}_{11}/\det \bar{G}_{11}$$

$$= \text{adj } \bar{G}_{11} = \text{adj } \bar{G},$$

since

$$\det \bar{G}_{11} = \det \bar{G} = \alpha - \beta - \gamma + \delta = 1.$$

In the $\bar{A}$-schema take $s_1 = 0$, $s_2 = 0$ and $t_1 = 0$, $t_2 = 0$ to get

$$u = \alpha\delta - \beta\gamma = v,$$

$$p_1 = \delta - \gamma \; (\geqq 0), \; p_2 = -\beta + \alpha \; (\geqq 0), \qquad \text{and}$$

$$q_1 = \delta - \beta \; (\geqq 0), \; q_2 = -\gamma + \alpha \; (\geqq 0).$$

These illustrate the Shapley-Snow formulas for a "basic solution": $G_{11} = G$ is basic kernel, $u = v = \det G/\det \bar{G}$, and the p's and q's are column-sums and row-sums from

$$\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} = \text{adj } G/\det \bar{G}.$$

If $\alpha > \beta, \alpha > \gamma, \delta > \beta, \delta > \gamma$ (termed "separated diagonals" [1], p. 173), then the optimal p's and q's above are all positive. In this case (an instance of a "completely mixed" game of I. Kaplansky [5]), G is the only basic kernel and the optimal p's and q's are unique. However, if at least one among the p's and q's is zero, then G is not the only basic kernel. For example, if $p_2 = -\beta + \alpha = 0$ (i.e., $\beta = \alpha$), then there is also a basic kernel of order one, i.e., a "saddlepoint," provided by the entry α in the payoff matrix, to which correspond the extreme optimal strategies

$$p_1 = 1, p_2 = 0 \qquad \text{and} \qquad q_1 = 1, q_2 = 0.$$

If $p_2 = -\beta + \alpha = 0$ and $q_2 = -\gamma + \alpha = 0$ (i.e., $\gamma = \beta = \alpha$), then these extreme optimal strategies are the *same* as those above arising from the basic kernel G of order two. In this case, there is still another basic kernel of order one (saddlepoint),

	$-v$	q_1	q_2	
u	0	-1	-1	$= -1$
p_1	1	α	β	$= -t_1$
p_2	1	γ	δ	$= -t_2$
	$=1$	$=s_1$	$=s_2$	

$\longrightarrow$

	1	t_1	t_2	
1	$\alpha\delta - \beta\gamma$	$\delta - \gamma$	$-\beta + \alpha$	$= v$
s_1	$-\delta + \beta$	1	-1	$= -q_1$
s_2	$\gamma - \alpha$	-1	1	$= -q_2$
	$=u$	$=p_1$	$=p_2$	

provided by the entry γ in the payoff matrix, to which correspond the extreme optimal strategies

$p_1 = 0, p_2 = 1$ and $q_1 = 1, q_2 = 0$.

• *Example 2*

As payoff matrix, take

$$G = \begin{bmatrix} 1 & -1 & 0 \\ -6 & 3 & -2 \\ 8 & -5 & 2 \end{bmatrix}.$$

Make the block-pivot transformation

	$-v$	q_2	q_3	q_1	
u	0	-1	-1	-1	$= -1$
p_1	1	-1	0	1	$= -t_1$
p_2	1	3	-2	-6	$= -t_2$
p_3	1	-5	2	8	$= -t_3$
	$=1$	$=s_2$	$=s_3$	$=s_1$	

↓

	1	t_1	t_2	q_1	
1	$-1/3$	5/6	1/6	1/6	$= v$
s_2	$-1/3$	$-1/6$	1/6	$-5/6$	$= -q_2$
s_3	$-2/3$	1/6	$-1/6$	11/6	$= -q_3$
p_3	0	-2	1	0	$= -t_3$
	$=u$	$=p_1$	$=p_2$	$=s_1$	

from the $\bar{G}$-schema (suitably permuted), above, to an $\bar{A}$-schema, below. Here the (optimal) block-pivot $\bar{G}_{11}$ is the three-by-three submatrix of $\bar{G}$ gotten by bordering

$$G_{11} = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}.$$

In the $\bar{A}$-schema take $s_2 = s_3 = p_3 = 0$ and $t_1 = t_2 = q_1 = 0$ to get

$u = -1/3 = v,$

$p_1 = 5/6, \quad p_2 = 1/6, \quad s_1 = 1/6$

and

$q_2 = 1/3, \quad q_3 = 2/3, \quad t_3 = 0.$

That is, the extreme optimal strategies corresponding to the above basic kernel G_{11} are

$p_1 = 5/6, \quad p_2 = 1/6, \quad p_3 = 0$

and

$q_1 = 0, \quad q_2 = 1/3, \quad q_3 = 2/3.$

Another block-pivot transformation

	$-v$	q_2	q_3	q_1	
u	0	-1	-1	-1	$= -1$
p_2	1	3	-2	-6	$= -t_2$
p_3	1	-5	2	8	$= -t_3$
p_1	1	-1	0	1	$= -t_1$
	$=1$	$=s_2$	$=s_3$	$=s_1$	

↓

	1	t_2	t_3	q_1	
1	$-1/3$	7/12	5/12	1/6	$= v$
s_2	$-1/3$	1/12	$-1/12$	$-5/6$	$= -q_2$
s_3	$-2/3$	$-1/12$	1/12	11/6	$= -q_3$
p_1	0	$-1/2$	$-1/2$	0	$= -t_1$
	$=u$	$=p_2$	$=p_3$	$=s_1$	

having

$$G_{11} = \begin{bmatrix} 3 & -2 \\ -5 & 2 \end{bmatrix}$$

as basic kernel, yields in similar fashion the game value

$u = -1/3 = v$

and the extreme optimal strategies

$p_1 = 0, \quad p_2 = 7/12, \quad p_3 = 5/12$

and

$q_1 = 0, \quad q_2 = 1/3, \quad q_3 = 2/3.$

Elementary pivot steps

A (combinatorial equivalence) transformation by a block-pivot of order r, as from a $\bar{G}$-schema to an $\bar{A}$-schema, exchanges r of the individual marginal labels (variable or constant) at the left with r labels at the bottom and the r parallel labels at the right with the r parallel labels at the top, signs being reversed in the latter exchange. Such a block-pivot transformation can always be decomposed into a succession of pivot transformations of order one, exchanging just one label on a margin at a time, together with suitable permutation of rows and/or columns at the end (see [2], Theorem 7).

An elementary pivot step (pivot transformation of order one) works as follows. (See E. Stiefel [6] for the connection with "Jordan elimination".)

$$\begin{array}{c|ccccc|l}
 & & \eta & & \eta' & & \\
\hline
 & & \vdots & & \vdots & & \\
\xi & \cdots & \alpha & \cdots & \beta & \cdots & = -\tau \\
 & & \vdots & & \vdots & & \\
\xi' & \cdots & \gamma & \cdots & \delta & \cdots & = -\tau' \\
 & & \vdots & & \vdots & & \\
\hline
 & & =\sigma & & =\sigma' & &
\end{array}$$

Take an entry $\alpha \neq 0$ as *pivot*. Solve for ξ and $-\eta$ the equations

$$\cdots + \xi\alpha + \cdots + \xi'\gamma + \cdots = \sigma$$

$$\cdots + \alpha\eta + \cdots + \beta\eta' + \cdots = -\tau$$

of the column and row containing the pivot α. Then substitute for ξ and η in the remaining column and row equations, such as

$$\cdots + \xi\beta + \cdots + \xi'\delta + \cdots = \sigma'$$

$$\cdots + \gamma\eta + \cdots + \delta\eta' + \cdots = -\tau'.$$

There result new column and row equation-systems described by the schema

$$\begin{array}{c|ccccc|l}
 & & \tau & & \eta' & & \\
\hline
 & & \vdots & & \vdots & & \\
\sigma & \cdots & \alpha^{-1} & \cdots & \alpha^{-1}\beta & \cdots & = -\eta \\
 & & \vdots & & \vdots & & \\
\xi' & \cdots & -\gamma\alpha^{-1} & \cdots & \delta - \gamma\alpha^{-1}\beta & \cdots & = -\tau' \\
 & & \vdots & & \vdots & & \\
\hline
 & & =\xi & & =\sigma' & &
\end{array}$$

Since an elementary pivot step is a special case ($r = 1$) of a block-pivot transformation, the pattern is the same as that in the block-pivot formulas previously given for $\bar{A}$'s submatrices in terms of $\tilde{G}$'s submatrices.

The formal rules for the above elementary pivot step may be summarized briefly as follows: Replace the pivot α ($\neq 0$) by $1/\alpha$. Multiply each remaining entry β in α's row by $1/\alpha$ and each remaining entry γ in α's column by $-1/\alpha$. Add to every other entry δ the product $(-\gamma/\alpha)\beta$ of the new entry $-\gamma/\alpha$ in δ's row and α's column and the old entry β in δ's column and α's row. Finally, exchange the marginal labels attached to α's row and column (with reversal of sign at top and right), but make no change in the other marginal labels.

In the following sections a simple constructive procedure, adapted from G. B. Dantzig [7], will be outlined for the determination of an optimal block-pivot $\tilde{G}_{11}$ and the corresponding $\bar{A}$-schema by a succession of elementary pivot steps, together with suitable permutation of rows and columns at the end. This provides an effective practical means of computing (extreme) optimal strategies by linear programming, more natural and direct than the well-known one that depends on the game value being positive [8] (also [4], p. 74).

Preliminary transformation to dual linear programs in a feasible form

For the purpose at hand, form the equation-systems

$$\begin{array}{rcl}
p_1 + \cdots + p_m &=& 1 \\
-u + p_1 g_{11} + \cdots + p_m g_{m1} &=& s_1 \\
\vdots & & \vdots \\
-u + p_1 g_{1n} + \cdots + p_m g_{mn} &=& s_n
\end{array}$$

and

$$\begin{array}{rcl}
-q_1 - \cdots - q_n &=& -1 \\
v - g_{11}q_1 - \cdots - g_{1n}q_n &=& t_1 \\
\vdots & & \vdots \\
v - g_{m1}q_1 - \cdots - g_{mn}q_n &=& t_m
\end{array}$$

described by the columns and rows of the $\tilde{G}$-schema

	v	$-q_1$	$-q_2$	$\cdots$	$-q_n$	
$-u$	0	1	1	$\cdots$	1	$= -1$
p_1	1	g_{11}	g_{12}	$\cdots$	g_{1n}	$= t_1$
p_2	1	g_{21}	g_{22}	$\cdots$	g_{2n}	$= t_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
p_m	1	g_{m1}	g_{m2}	$\cdots$	g_{mn}	$= t_m$
	$=1$	$=s_1$	$=s_2$	$\cdots$	$=s_n$	

.

These equation-systems differ from those described by the columns and rows of the $\tilde{G}$-schema (at the end of the Introduction) only in the inessential fact that all equations, except the first, in the row system have been multiplied through by minus one.

Formally, the $\tilde{G}$-schema is gotten from the $\bar{G}$-schema (1) by reversing the signs of all marginal labels at the top and right and (2) by multiplying through the entries in the first row by minus one and reversing the signs of its two marginal labels. Note that (1) makes no change *within* the schema but that (2) does, although this change reflects only a trivial alteration in the equation-systems described by the schema. (The change from $\bar{G}$ to $\tilde{G}$ is made to fit the following computational procedure to the pattern of the Dantzig "simplex method." This change, convenient but not essential, does emphasize that there is some arbitrariness in the precise schema chosen for a given purpose.)

Solve the first column equation in the $\tilde{G}$-schema for p_m and the last row equation for $-v$, and substitute for p_m and v in the remaining equations of the two systems. There results the following $\tilde{G}'$-schema

$$
\begin{array}{c|ccccc|l}
 & -t_m & -q_1 & -q_2 & \cdots & -q_n & \\
\hline
-u & 0 & 1 & 1 & \cdots & 1 & = -1 \\
\hdashline
p_1 & -1 & g'_{11} & g'_{12} & \cdots & g'_{1n} & = t_1 \\
p_2 & -1 & g'_{21} & g'_{22} & \cdots & g'_{2n} & = t_2 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\hdashline
1 & 1 & g_{m1} & g_{m2} & \cdots & g_{mn} & = -v \\
\hline
 & = p_m & = s_1 & = s_2 & \cdots & = s_n &
\end{array}
$$

where

$$g'_{ij} = g_{ij} - g_{mj} \qquad (i \neq m).$$

This is an elementary pivot step, using as pivot the entry 1 in the lower left corner of the $\tilde{G}$-schema.

Now solve the last column equation in the $\tilde{G}'$-schema for $-u$ and the first row equation for q_n, and substitute for $-u$ and $-q_n$ in the remaining equations of the two systems. There results the following $\tilde{G}''$-schema

$$
\begin{array}{c|cccc:c|l}
 & -t_m & -q_1 & -q_2 & \cdots & 1 & \\
\hline
s_n & 0 & 1 & 1 & \cdots & 1 & = q_n \\
p_1 & -1 & g''_{11} & g''_{12} & \cdots & -g'_{1n} & = t_1 \\
p_2 & -1 & g''_{21} & g''_{22} & \cdots & -g'_{2n} & = t_2 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\hdashline
1 & 1 & g'_{m1} & g'_{m2} & \cdots & -g_{mn} & = -v \\
\hline
 & = p_m & = s_1 & = s_2 & \cdots & = -u &
\end{array}
$$

where

$$g''_{ij} = g'_{ij} - g'_{in} = g_{ij} - g_{mj} - g_{in} + g_{mn}$$

and

$$g'_{mj} = g_{mj} - g_{mn} \qquad (i \neq m,\ j \neq n).$$

This is also an elementary pivot step, using as pivot the entry 1 in the upper right corner of the $\tilde{G}'$-schema. Note the invariance of the entries g'' and g' when each payoff g_{ij} is replaced by $g_{ij} + k$, where k is an arbitrary constant; the only change in the entire $\tilde{G}''$-schema is that the lower right corner entry $-g_{mn}$ is replaced by $-g_{mn} - k$.

By a mere change of notation, rewrite the $\tilde{G}''$-schema as the following $\tilde{A}^0$-schema:

$$
\begin{array}{c|cccc:c|l}
 & -y_n^0 & -y_1^0 & -y_2^0 & \cdots & 1 & \\
\hline
x_m^0 & a_{mn}^0 & a_{m1}^0 & a_{m2}^0 & \cdots & b_m^0 & = v_m^0 \\
x_1^0 & a_{1n}^0 & a_{11}^0 & a_{12}^0 & \cdots & b_1^0 & = v_1^0 \\
x_2^0 & a_{2n}^0 & a_{21}^0 & a_{22}^0 & \cdots & b_2^0 & = v_2^0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\hdashline
1 & -c_n^0 & -c_1^0 & -c_2^0 & \cdots & -d^0 & = -v \\
\hline
 & = u_n^0 & = u_1^0 & = u_2^0 & \cdots & = -u &
\end{array}
$$

or

$$
\begin{array}{c|c:c|l}
 & -Y^0 & 1 & \\
\hline
X^0 & A^0 & B^0 & = V^0 \\
\hdashline
1 & -C^0 & -d^0 & = -v \\
\hline
 & = U^0 & = -u &
\end{array}
.
$$

This $\tilde{A}^0$-schema pertains to dual linear programs in the following form:

Minimize $-u = X^0B^0 - d^0$ constrained by

$$U^0 = X^0A^0 - C^0 \geqq 0,\ X^0 \geqq 0.$$

Maximize $-v = C^0Y^0 - d^0$ constrained by

$$V^0 = -A^0Y^0 + B^0 \geqq 0,\ Y^0 \geqq 0.$$

$Y^0 = 0$, $V^0 = B^0$, $v = d^0$ is a feasible solution of the second program (corresponding to player *II*'s problem) if, and only if, $B^0 \geqq 0$. This condition can be satisfied merely by arranging that the payoff entry g_{mn} is maximal in its column, i.e., $g_{mn} \geqq g_{in}$ for $i = 1, \cdots, m - 1$. Thus it may (and will) be assumed that the second program associated with the $\tilde{A}^0$-schema has the feasibility property $B^0 \geqq 0$, to be utilized in the remaining part of the game-solution procedure.

Note that the $\tilde{A}^0$-schema can be reformulated as the $\bar{A}^0$-schema

	1	Y^0	
1	d^0	$-C^0$	$= v$
X^0	$-B^0$	A^0	$= -V^0$,
	$= u$	$= U^0$	

gotten from the $\tilde{G}$-schema via the block-pivot

$$\tilde{G}_{11} = \begin{bmatrix} 0 & -1 \\ 1 & g_{mn} \end{bmatrix}$$

of order two. The block-pivot $\tilde{G}_{11}$ is *optimal* if, and only if, g_{mn} is a *saddlepoint* (i.e., g_{mn} is minimal in its row of the payoff matrix G, as well as maximal in its column). This suggests that a Shapley-Snow basic kernel G_{11} of order greater than one may be regarded as a generalized saddlepoint. It might be called a "saddle-block."

• *Example*

Here the payoff matrix G is the same as in Example 2. Pivot entries are starred.

	v	$-q_1$	$-q_2$	$-q_3$	
$-u$	0	1	1	1	$= -1$
p_1	1	1	−1	0	$= t_1$
p_2	1	−6	3	−2	$= t_2$
p_3	1*	8	−5	2	$= t_3$
	$= 1$	$= s_1$	$= s_2$	$= s_3$	

($\tilde{G}$-schema)

↓

	$-t_3$	$-q_1$	$-q_2$	$-q_3$	
$-u$	0	1	1	1*	$= -1$
p_1	−1	−7	4	−2	$= t_1$
p_2	−1	−14	8	−4	$= t_2$
1	1	8	−5	2	$= -v$
	$= p_3$	$= s_1$	$= s_2$	$= s_3$	

($\tilde{G}'$-schema)

↓

	$-t_3$	$-q_1$	$-q_2$	1	
s_3	0	1	1	1	$= q_3$
p_1	−1	−5	6	2	$= t_1$
p_2	−1	−10	12	4	$= t_2$
1	1	6	−7	−2	$= -v$
	$= p_3$	$= s_1$	$= s_2$	$= -u$	

($\tilde{G}''$-schema)

The $\tilde{G}''$-schema (or $\tilde{A}^0$-schema) has the desired "feasible form": take $t_3 = q_1 = q_2 = 0$ to get $q_3 = 1$, $t_1 = 2$, $t_2 = 4$, all nonnegative! This is because the entry 2 in the lower right corner of the $\tilde{G}$-schema is maximal in its column of the payoff matrix G.

Note that a like result is gotten from the following block-pivot transformation (of order two) from a $\tilde{G}$-schema to an $\bar{A}^0$-schema:

	$-v$	q_3	q_1	q_2	
u	0	−1	−1	−1	$= -1$
p_3	1	2	8	−5	$= -t_3$
p_1	1	0	1	−1	$= -t_1$
p_2	1	−2	−6	3	$= -t_2$
	$= 1$	$= s_3$	$= s_1$	$= s_2$	

↓

	1	t_3	q_1	q_2	
1	2	1	6	−7	$= v$
s_3	−1	0	1	1	$= -q_3$
p_1	−2	−1	−5	6	$= -t_1$
p_2	−4	−1	−10	12	$= -t_2$
	$= u$	$= p_3$	$= s_1$	$= s_2$	

Completion of game solution by simplex method

From the $\tilde{A}^0$-schema with $B^0 \geqq 0$, as specified above, the solution procedure follows the usual technique of the Dantzig "simplex method" (along the lines of the elementary version used by S. Vajda [9]). Moving through a succession of elementary pivot steps, with resulting schemata

$$
\begin{array}{c|c:c|l}
 & -Y^r & 1 & \\
\hline
X^r & A^r & B^r & = V^r \\
\hdashline
1 & -C^r & -d^r & = -v \\
\hline
 & = U^r & = -u &
\end{array},
$$

the goal is to drive $-C^r \geqq 0$ while keeping $B^r \geqq 0$. In each $\tilde{A}^r$-schema ($r = 0, 1, \cdots$) select a column with its $-c_j^r < 0$ and then, from among the entries $a_{ij}^r > 0$ in this column, choose as *pivot* an entry a_{ij}^r for which b_i^r/a_{ij}^r is minimal. With this pivot, make the elementary pivot step to an $\tilde{A}^{r+1}$-schema. The rule for choice of the pivot a_{ij}^r ensures that $B^{r+1} \geqq 0$ and that $-d^{r+1} \geqq -d^r$.

Since there are only a finite number of possible $\tilde{A}^r$-schemata and the column and row equation-systems are feasible at each step (the equivalent systems of $\tilde{G}$ being feasible), *either* the process terminates in a final $\tilde{A}$-schema

$$
\begin{array}{c|c:c|l}
 & -Y & 1 & \\
\hline
X & A & B & = V \\
\hdashline
1 & -C & -d & = -v \\
\hline
 & = U & = -u &
\end{array},
$$

having

$B \geqq 0$ and $-C \geqq 0$,

or the process "cycles," i.e., there occurs an $\tilde{A}^s$-schema which is the same as a previous $\tilde{A}^r$-schema (except for possible permutation of rows and/or columns). Since $-d^{r+1} \geqq -d^r$, this rare event can happen only if $d^r = d^{r+1} = \cdots = d^s$. However, such "cycling" can be avoided (see an inductive proof by G. B. Dantzig [10] in a companion paper in this issue of the *IBM Journal*). From the final $\tilde{A}$-schema read off the (extreme) optimal solutions

$X = 0, U = -C$ and $Y = 0, V = B$

yielding $u = d = v$.

By suitable permutations of rows and columns, rearrange the initial $\tilde{G}$-schema and the final $\tilde{A}$-schema

$$
\begin{array}{c|c:cc|l}
 & v & -Q_1 & -Q_2 & \\
\hline
-u & 0 & 1 & 1 & = -1 \\
\hdashline
P_1 & 1 & G_{11} & G_{12} & = T_1 \\
P_2 & 1 & G_{21} & G_{22} & = T_2 \\
\hline
 & = 1 & = S_1 & = S_2 &
\end{array}
$$

and

$$
\begin{array}{c|cc:c|l}
 & -Y_1 & -Y_2 & 1 & \\
\hline
X_1 & A_{11} & A_{12} & B_1 & = V_1 \\
X_2 & A_{21} & A_{22} & B_2 & = V_2 \\
\hdashline
1 & -C_1 & -C_2 & -d & = -v \\
\hline
 & = U_1 & = U_2 & = -u &
\end{array},
$$

so that

$X_1 = S_1, \quad X_2 = P_2; \quad U_1 = P_1, \quad U_2 = S_2$

and

$Y_1 = T_1, \quad Y_2 = Q_2; \quad V_1 = Q_1, \quad V_2 = T_2.$

Then it can be shown that the corresponding $\tilde{A}$-schema is obtained from the corresponding $\tilde{G}$-schema by the block-pivot transformation

$$
\begin{array}{c|c:c|c|l}
 & -v & Q_1 & Q_2 & \\
\hline
u & 0 & -1 & -1 & = -1 \\
\hdashline
P_1 & 1 & G_{11} & G_{12} & = -T_1 \\
\hline
P_2 & 1 & G_{21} & G_{22} & = -T_2 \\
\hline
 & = 1 & = S_1 & = S_2 &
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{c|c:c|c|l}
 & 1 & T_1 & Q_2 & \\
\hline
1 & d & -C_1 & -C_2 & = v \\
\hdashline
S_1 & -B_1 & A_{11} & A_{12} & = -Q_1. \\
\hline
P_2 & -B_2 & A_{21} & A_{22} & = -T_2 \\
\hline
 & = u & = P_1 & = S_2 &
\end{array}
$$

Thus, an optimal block-pivot $\tilde{G}_{11}$ has been determined, with G_{11} as Shapley-Snow basic kernel and $P_1 = -C_1, P_2 = 0$ and $Q_1 = B_1, Q_2 = 0$ as extreme optimal strategies, the game value being $u = d = v$.

• *Completion of previous example*

Starting from the $\tilde{G}''$-schema of the previous Example, proceed by elementary pivot steps, as follows. Pivot entries are starred.

	$-t_3$	$-q_1$	$-q_2$	1	
s_3	0	1	1	1	$= q_3$
p_1	−1	−5	6*	2	$= t_1$
p_2	−1	−10	12	4	$= t_2$
1	1	6	−7	−2	$= -v$
	$=p_3$	$=s_1$	$=s_2$	$= -u$	

($\tilde{A}^0$-schema)

↓

	$-t_3$	$-q_1$	$-t_1$	1	
s_3	1/6	11/6	−1/6	2/3	$= q_3$
s_2	−1/6	−5/6	1/6	1/3	$= q_2$
p_2	1*	0	−2	0	$= t_2$
1	−1/6	1/6	7/6	1/3	$= -v$
	$=p_3$	$=s_1$	$=p_1$	$= -u$	

($\tilde{A}^1$-schema)

↓

	$-t_2$	$-q_1$	$-t_1$	1	
s_3	−1/6	11/6	1/6	2/3	$= q_3$
s_2	1/6	−5/6	−1/6	1/3	$= q_2$
p_3	1	0	−2	0	$= t_3$
1	1/6	1/6	5/6	1/3	$= -v$
	$=p_2$	$=s_1$	$=p_1$	$= -u$	

($\tilde{A}$-schema)

Taking $s_3 = s_2 = p_3 = 0$ and $t_2 = q_1 = t_1 = 0$, read off the game value $u = -1/3 = v$ and the extreme optimal strategies $p_1 = 5/6$, $p_2 = 1/6$, $p_3 = 0$ and $q_1 = 0$, $q_2 = 1/3$, $q_3 = 2/3$.

Thus, a succession of elementary pivot steps has achieved the same result as the first block-pivot transformation in Example 2, above. Similarly, the result of the second block-pivot transformation in Example 2 can be achieved by returning to the $\tilde{G}''$-schema and making the following elementary pivot step.

	$-t_3$	$-q_1$	$-q_2$	1	
s_3	0	1	1	1	$= q_3$
p_1	−1	−5	6	2	$= t_1$
p_2	−1	−10	12*	4	$= t_2$
1	1	6	−7	−2	$= -v$
	$=p_3$	$=s_1$	$=s_2$	$= -u$	

($\tilde{A}^0$-schema)

↓

	$-t_3$	$-q_1$	$-t_2$	1	
s_3	1/12	11/6	−1/12	2/3	$= q_3$
p_1	−1/2	0	−1/2	0	$= t_1$
s_2	−1/12	−5/6	1/12	1/3	$= q_2$
1	5/12	1/6	7/12	1/3	$= -v$
	$=p_3$	$=s_1$	$=p_2$	$= -u$	

($\tilde{A}$-schema)

Taking $s_3 = p_1 = s_2 = 0$ and $t_3 = q_1 = t_2 = 0$, read off the game value $u = -1/3 = v$ again and the extreme optimal strategies $p_1 = 0$, $p_2 = 7/12$, $p_3 = 5/12$ and $q_1 = 0$, $q_2 = 1/3$, $q_3 = 2/3$.

Comment

A block-pivot transformation from a $\bar{G}$-schema to an $\bar{A}$-schema and a succession of elementary pivot steps from a $\tilde{G}$-schema to an $\tilde{A}$-schema, as presented in this paper, demonstrate operationally the efficacious methods, theoretical and practical, of "combinatorial equivalence." These methods apply generally in the combinatorial linear algebra of linear inequalities and related systems. They pertain to the matrix representation of the abstract linear-transformation structure of a partially-ordered vector space over an ordered field (in which the "nonnegative orthant" of vectors $X \geqq 0$ conforms to the nonnegative halfline of scalars $x \geqq 0$ in the ordered field).

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