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# Extension of Moore-Shannon Model for Relay Circuits

**Abstract: The Moore-Shannon model for switching circuits is extended to show how the number of redundant relays needed to improve reliability depends on the logical function of the entire circuit. The reliabilities of AND, OR, and EXCLUSIVE-OR relay circuits are studied as a function of the number of relays, the network topology, and the distribution of inputs. For the case of intermittent failures, a procedure is developed for calculating the reliability of combinational switching circuits, defined as the probability that the circuit will function as specified, averaged over all possible inputs, and subject to the idealizing assumptions of the Moore-Shannon model. The redundancies required to achieve a specified increase in reliability, although considerably smaller than for alternative methods, are still enormous. It is shown that a good way to improve an AND circuit, for example, is to use a series-parallel network in which the number of parallel lines varies with the logarithm of the number of basic AND circuits connected in series to form each line.**

### *Introduction*

Combinational switching circuits which can be found in the arithmetic and control sections of digital computers, telephone exchanges, and other control centers, tend to be unreliable when the number of constituent switches becomes very large. Thus, an average time of about 10 hours between breakdowns is perhaps a reasonable figure for a large digital computer. A theoretically interesting and practically important question is to determine the possibility of substituting a given switching circuit by another switching circuit with more switches, but with the same number of inputs, outputs, and the same behavior, and with as low a failure probability as desired. After a precise formulation of this problem in terms of the Moore-Shannon-von Neumann model for relay circuits,[1,2] it becomes a purely mathematical problem of determining sufficient conditions on the structure of several proposed nets to replace a given circuit such that its probability of error, suitably defined, becomes arbitrarily small for a sufficiently large redundancy.

One such network is a replication of the given circuit, with each switch replaced by a sufficiently large hammock network which acts like a single switch of any specified reliability. Moore and Shannon[1] have proved that this can be done with relays of arbitrarily poor reliability by simply using enough of them in the right way. They also showed that in a hammock network it takes at least $[(\log b)/(\log a)]^2$ relays of error probability $a$, where $a<\frac{1}{4}$, for the network to function as a relay with error probability $b$, where $b\ll a$. For example, suppose that an AND circuit is to be designed which will fail at most once in half-a-million cases, on the average. This design would consist of two relay contacts in series connection, in which either relay contact would fail at most once in a million cases, on the average, with the conservative assumption that the circuit fails when a single relay fails. If the only relays which are available, however, have failure rates of 0.005, then we require at least nine such relays, connected in a $3\times3$ hammock network with a common coil, to replace each of the two relays in series. Thus, 18 relays of the above type are required to produce such a circuit. If, however, we do not assume that the circuit necessarily fails when a single relay fails, then we might be able to produce such a circuit with fewer than 18 relays of the same type.

In this paper, the results of Moore-Shannon are extended by showing that knowledge of the intended logical function of the entire circuit leads to economies in the redundant use of relays to improve reliability. In doing this, we have defined the *reliability* of a circuit as the *probability that it* functions as specified, averaged over all possible inputs. Both analytical and computational methods for calculating this quantity for a large class of circuits are described. Using these methods, we examine several redundant circuits and estimate the redundancies required to achieve a desired reliability for the case of intermittent failures. The significant findings are that increasing the redundancy will not give arbitrarily high reliability, except under special conditions on the proportions of the network, and that these conditions have a very simple and interesting form.

## The model

The starting point for our analysis is the Moore-Shannon model for switching circuits.[1] It should be remembered that we are dealing, not with a description of switches of any particular *physical* nature, but with a model of switching circuits involving many idealizations from actual devices. Although electromechanical relays are probably the switches which this model describes best, for some purposes, cryotrons, magnetic cores, and other switches are not grossly misrepresented by the assumptions; but this is irrelevant for the exploration of the formal model. To introduce the notation and emphasize the limitations to which all the following results are subject, it will be helpful to summarize the main assumptions of this model in its extension to our problem.

(1) A switch (any operable bistable device) is regarded as consisting of two parts, the "contact" and the "coil."* The contact is either open or closed, and the coil either is or is not energized. All four combinations of coil and contact states are considered possible.

(2) If more than one contact is served by the same coil, it is assumed that all coils are in the same state. In an OR circuit, for instance, all the resistors, capacitors, and inductors should be such as to cause a current in either branch to be above the threshold to be called a "1" at the output. These circuit parameters are assumed to remain consistently at fixed values.

(3) Only failures at the contacts are taken into account. This excludes many common errors in the coils, such as breaking of lines, loosening of solder joints, short circuits of the coil and contact, et cetera.

(4) The mechanism which causes a relay to fail at any time is independent of the relay's prior history and past failures and use. That is, wear, fatigue, and catastrophic failure (e.g., a relay permanently welded shut) are excluded; only *intermittent* failures are considered. Thus, the behavior of a switch is supposed to be completely described by the following two conditional probabilities:

$$\begin{aligned} a &= P\,(\text{contact is closed}|\text{coil is energized}) \\ c &= P\,(\text{contact is closed}|\text{coil is not energized}) \end{aligned} \qquad (1)$$

(5) The final assumption on which our model is built takes all the relays to be statistically independent. This is perhaps the most severe idealization in this context, since the cause for failure of one switch (e.g., dirt, moisture, temperature) may also cause at least the neighboring relays to fail at about the same time. For convenience of notation we shall also make the (nonessential) assumption that all relays are characterized by the same parameters $a$ and $c$.

Unless stated otherwise, we shall regard all the relays studied here as ideally normally open, which means that $a>c$. This means that for a perfect relay, $a=1$ and $c=0$.

## Basic AND and OR circuits

In order to introduce our definition of circuit reliability with mathematical precision, it will be useful to start with AND circuits and OR circuits. Furthermore, in view of the fact that a circuit with any logical function can be built in terms of OR circuits, AND circuits, and normally closed relays exclusively, the analysis of these special cases is of central importance.

A simple way to construct an $n$-way AND circuit—that is, a circuit which conducts current if and only if all $n$ input coils are energized—is to connect $n$ normally open relays in series. To construct an OR circuit—which is open if and only if all $n$ input coils are unenergized—is to construct the dual of the AND circuit, or $n$ normally open relays connected in parallel.

We shall suppose that the $n$ states of the $n$ input coils are statistically independent, and $p$ the probability that a given input coil is energized. There are obviously $2^n$ possible input configurations, with $\binom{n}{k}p^k(1-p)^{n-k}$ being the probability of all input configurations in which exactly $k$ of the $n$ coils are energized. An AND circuit should be open for all input configurations in which $k\neq n$. The conditional probability that $n$ independent relays in series form an open circuit, given one of the above input distributions with $k\neq n$, is simply $1-a^kc^{n-k}$, because $a^kc^{n-k}$ is the corresponding probability of a closed circuit. When $k=n$, we seek the conditional probability that $n$ relays in series form a closed circuit, or that all the $n$ independent relays are closed, given that all coils are energized. This is simply $a^n$. Each of the $n$ terms so generated, when multipled by $\binom{n}{k}p^k(1-p)^{n-k}$, $k=0, 1,\cdots,n$ gives the joint probability of an input configuration with $n$ coils energized and a correct response from the circuit as a conjunction of $n$ variables. To obtain the probability that the circuit functions correctly, we sum over all inputs, and obtain:

$$R=\sum_{k=0}^{n-1}\binom{n}{k}p^kq^{n-k}(1-a^kc^{n-k})+p^na^n$$

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*The abstract terms "contact" and "coil" refer to the storage and gating functions of any switching device.

$$=\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}-\sum_{k=0}^{n}\binom{n}{k}(pa)^k(qc)^{n-k}$$

$$+p^na^n-p^n(1-a^n)$$

$$=1-(pa+qc)^n-p^n(1-2a^n)\ ,$$

$$\text{where } q=1-p\ . \qquad (2)$$

We shall use $R$ as a measure of reliability of the circuit as an AND circuit. It is clearly a function of $a$, $c$, $p$ and $n$. This measure treats all errors as equally important. This seemingly plausible assumption leads to the possibly objectionable result that, when $n$ is very large, $R$ is very close to 1 (in fact, if $p\neq 1$, then $\lim_{n\to\infty} R=1$), implying that an open circuit or no circuit at all would function almost as reliably as an AND circuit of $n$ relays in series. This follows directly from the nature of an AND circuit and the fact that the input configuration with all coils energized is a rare event when $n$ is large. It is clear that if this input is very likely, in particular if $p>1-(1/n)$, then an open circuit is no longer comparable to $n$ relays in series. Also, if the cost of circuit failure when all coils are energized is very high, this measure of reliability is misleading. If the costs of various errors could be estimated, then the expected loss of using the circuit would be a more suitable measure of *un*reliability. For example, if it were known that a failure of $n$ relays in series to produce an open circuit, when not all coils are energized, is $u$, and the cost of failure to close when all coils are energized is $v$, then the expected loss is

$$\sum_{k=0}^{n-1}\binom{n}{k}p^kq^{n-k}a^kc^{n-k}u+p^n(1-a^n)v$$

$$=u[(pa+qc)^n-p^na^n]+vp^n(1-a^n)\ .$$

Note that if $p<1$, $a\neq c$, this converges to 0 for any $u$, $v$, $a$ and $c$, as $n\to\infty$. In the absence of realistic estimates for such costs, the definition of reliability exemplified by Eq. (2) will be accepted, and explored for its mathematical properties. Finally, it should be emphasized that our measure of reliability concerns only the restricted case of an intermittent circuit failure in a single operation.

From the fact that the OR and AND circuits are completely dual, we obtain an important relation between $R(a,c,n,p)$, the reliability of $n$ relays in series as an AND circuit, and $R'(a,c,n,p)$, the reliability of $n$ relays in parallel as an OR circuit.

**Theorem 1.** $R'(a,c,n,p)=R(1-c,1-a,n,q)$ .

*Proof:*

$$R'(a,c,n,p)=q^n(1-c)^n+\sum_{k=1}^{n}\binom{n}{k}p^kq^{n-k}$$

$$\times[1-(1-a)^k(1-c)^{n-k}]\ .$$

The first term represents the joint probability that no coil is energized and that the $n$ relays in parallel are all open. The $k^{\text{th}}$ term in the sum is the joint probability that $k$ of the coils are energized and that not all of the $n$ relays in parallel are open. After some simple algebraic steps, it is readily seen that

$$R'(a,c,n,p)=1-[p(1-a)+q(1-c)]^n-q^n[1-2(1-c)^n]\ . \qquad (3)$$

If we substitute $1-c$, $1-a$ and $p$ for $a$, $c$ and $q$, respectively, wherever the latter appear in Eq. (3), we obtain Eq. (2), which proves the theorem.

To obtain some feeling for how $R$ and $R'$ vary as functions of $a$ and $c$, consider the families of surfaces shown in Fig. 2 for $p=\frac{1}{2}$.

The point $(a,c)=(1,0)$ is an invariant on both surfaces. The intersection of both surfaces with the plane $a=1-c$ is the curve: $R=1-2^{-n+1}(1-a^n)$. Both surfaces approach the uppermost plane of the unit cube, $R=1$, except for one point, where $R=0$ as $n\to\infty$; for $R$, this discontinuity occurs at $a=c=1$; for $R'$, it occurs at $a=c=0$.

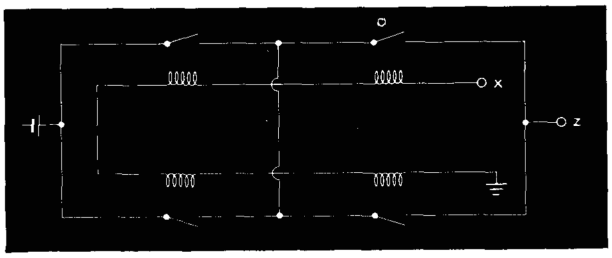
To get some understanding how $R$ and $R'$ vary with $p$ and $c$, it will be convenient to let $a=1-c$ (an assumption which will be made henceforth without explicit reiteration). The results are shown graphically in Fig. 3.

If $p=0$, then $R=1-c^n$ and $R'=(1-c)^n$. If $p=1$, then $R=a^n=(1-c)^n$ and $R'=1-(1-a)^n=1-c^n$. If $c=p=0$ or if $a=p=1$, we have perfect reliability, $R=R'=1$.

### Improved series AND circuits

As mentioned in the Introduction, we can make an AND circuit as reliable as we wish by connecting $n$ "relays" in series, but making each "relay" sufficiently reliable by substituting for it a network of less reliable relays according to the Moore-Shannon technique. In particular, we consider the network shown in Fig. 1; suppose that each of its four relays are replaced by a network just like that in Fig. 1; each of the 16 relays in the resulting network is also replaced by the same circuit, et cetera, until a network like Fig. 1 has been substituted for a single initial relay $m$ times. The coils of the $4^m$ contacts in this final circuit are connected in series, so that the entire circuit acts like a single

*Figure 1* **Hammock network (2 X 2) to replace a single relay.**



relay. Let $a_m$ be the probability that this "$m$-fold composed" network is closed when the coil is energized, and $c_m$ the probability that it is closed when the coil is not energized. We interpret $m=0$ to represent a single, basic relay, with $a_0=a$, $c_0=c$; $m=1$ represents the network of Fig. 1 in place of the single, basic relay, et cetera. It is easily shown[1] that

$$a_{m+1}=a_m^2(2-a_m)^2$$

and

$$c_{m+1}=c_m^2(2-c_m)^2 \; . \tag{4}$$

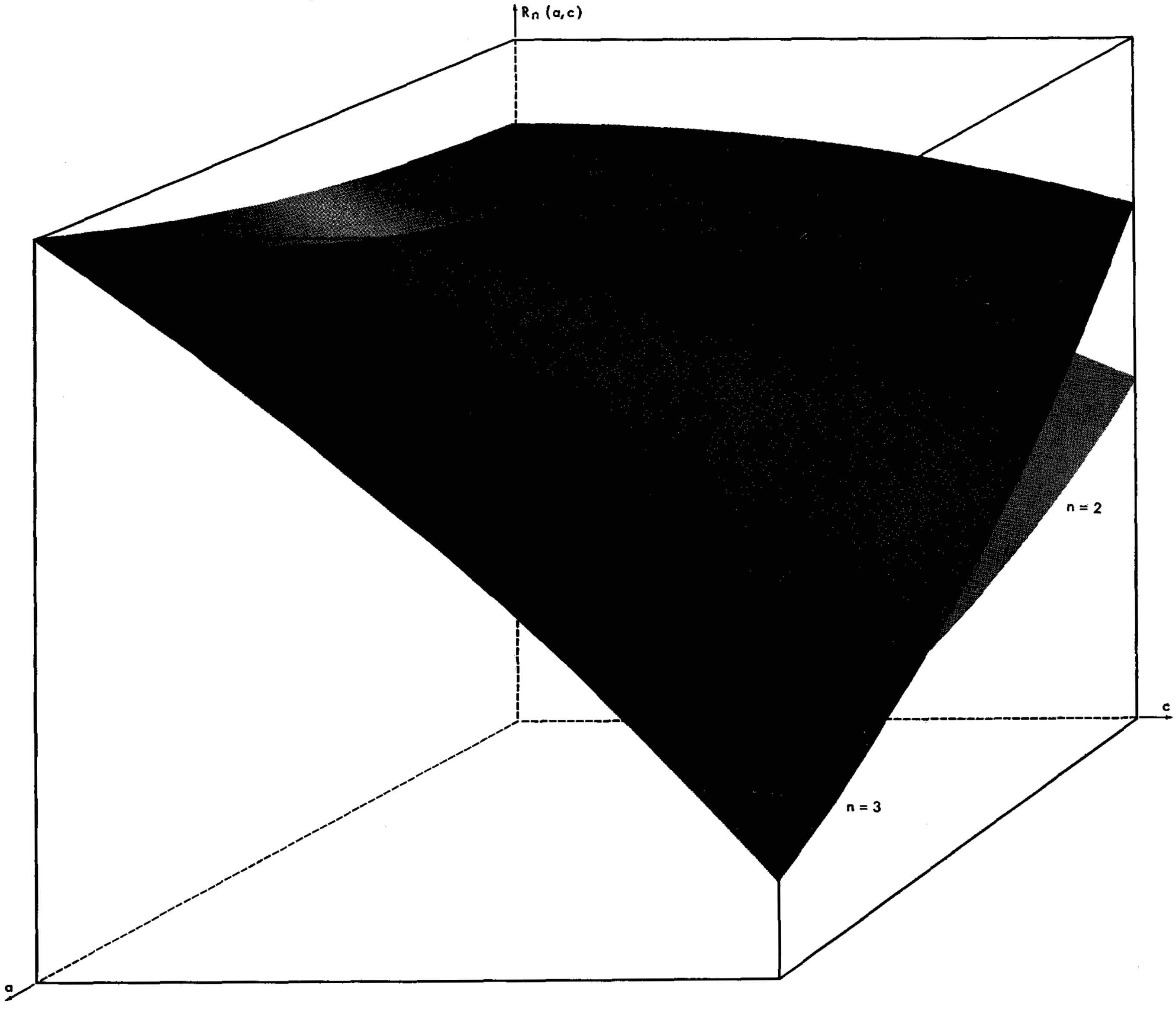
We now consider the reliability of $n$ such $m$-fold composed relays in series as an AND circuit in the manner described previously. To simplify the algebra, we shall take $p=\frac{1}{2}$ and $a=1-c$ throughout this section. Then, according to Eq. (2)

$$R_m=1-2^{-n}[(a_m+c_m)^n-2a_m^n+1]\equiv 1-Q_m \; . \tag{5}$$

We wish to determine the smallest integral value of $m$ such that the probability of error, $Q_m$, is less than or equal to any specified positive number $\epsilon$ for any given $n$ and $c$. We shall then show that this $m$ is smaller than the corresponding redundancy when the circuit logic is not taken into account. We shall have occasion to use the fact that for high-quality, normally open relays, $c$ is bounded by a number $d$, with $d \doteq 10^{-4}$ being a reasonable figure.

Because of the difficulty of obtaining an explicit formula for $Q_m$ in terms of $m$, $n$ and $c$ and of solving an equation like $Q_m=\epsilon$ for $m$, we shall, instead, look for two numbers, $m_1$ and $m_2$ with the following properties:

*Figure 2a* **Circuit reliability as a function of element reliabilities.** *Conjunction*



(1) There is no $m$ less than $m_1$, such that $Q_m \leq \epsilon$.

(2) The value of $m$ such that $Q_m \leq \epsilon$, need not exceed $m_2$. If $m = m_0$ represents a solution to the equation $Q_m = \epsilon$, then $m_1 \leq m_0 \leq m_2$, and the smaller $m_2 - m_1$, the better the bounds.

In the Appendix, it is demonstrated that $Q_m$ is a monotonically decreasing function of $m$, so that $m \geq m_1$ implies that $Q_m \leq \epsilon$. To obtain $m_1$, which is a lower bound on $m$, we seek a function, $Q_m'$, which is a lower bound on $Q_m$ and let $m = m_1$ be the solution of $Q_m' = \epsilon$.

**Theorem 2.** *Let*

$$Q_m' = 2^{-n}\left[nx(\cosh n^2x^2 - \sinh nx) - \frac{n}{2}(2c)^{2^m} - ny\right], \qquad (6)$$

*where*

$$x = [(2-d^2)c]^{2^m}/(2-d)^2$$

*and*

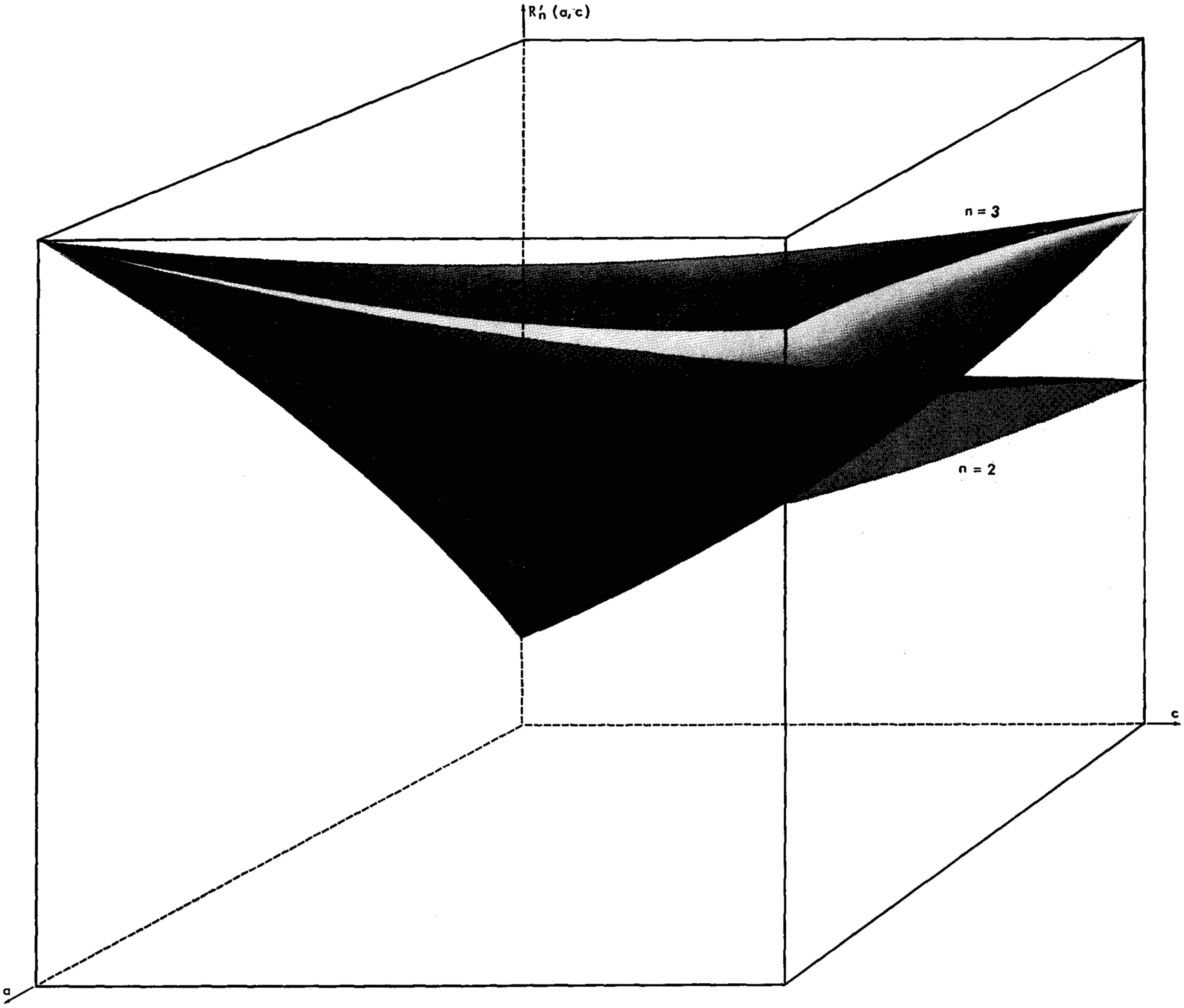
$$y = [(2-d)^2c]^{2^m}/(2-d)^2 \ .$$

*If* $c \leq d < \frac{1}{4}$, *then* $Q_m' \leq Q_m$ *for all* $m$.

The proof of this result contains a number of interesting mathematical techniques which may prove useful in a similar analysis. Since it is lengthy and represents a digression from the main theme of this section, it will be found in the Appendix.

To obtain $m_2$, which is an upper bound on $m$, we seek a function $Q_m''$ which bounds $Q_m$ from above, and let $m = m_2$ be the solution of $Q_m'' = \epsilon$. This means that there is an $m$ not exceeding $m_2$ such that $Q_m \leq \epsilon$.

*Figure 2b* **Circuit reliability as a function of element reliabilities.** *Disjunction*



**Theorem 3.** *Let*

$$Q_m'' = 2^{-n}n[(2c)^{2^m} - z(\cosh n^2 z - \sinh nz)] \qquad (7)$$

*where*

$$z = [(2-d^2)c]^{2^m}/(2-d^2) - \tfrac{1}{4}(4c)^{2^m} .$$

*Then* $Q_m'' \geq Q_m$ *for all* $m$.

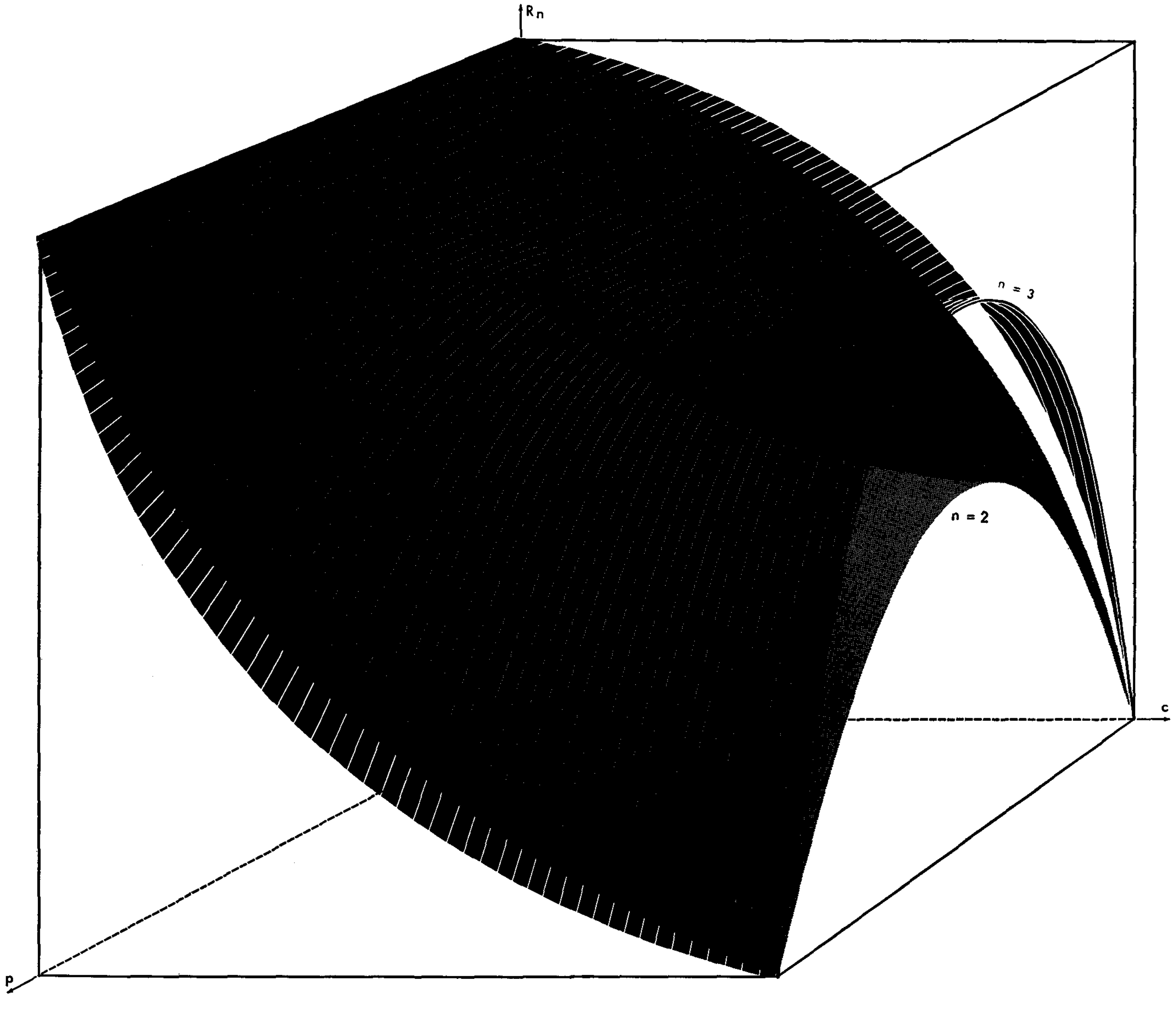
*Proof:* See Appendix.

*Example:* Suppose that we wanted to keep the probability of error below $\epsilon = 2^{-62}$, and we wanted a 32-way AND circuit, using relays with error probabilities of $c = 2^{-10}$. If we take $m = 2$, then, according to Theorems 2 and 3, a circuit with a total of 512 relays has a probability of error between $2^{-63}$ and $2^{-61}$. To achieve $\epsilon = 2^{-62}$ without taking the circuit logic into account, we let $r$ be the reliability of one of the $n$ "$m$-fold composed relays" in series and set $1-\epsilon = r^n$. Hence, $r \doteq 1-\epsilon/n$. According to Ref. 1, $m$ must be at least $\{[\lg(\epsilon/n)]/\lg c\}^2$, or $(67/10)^2 \doteq 45$ (lg will denote $\log_2$). Thus, at least $32 \times 45 = 1440$ relays are required, which is almost triple the number required when the circuit logic is taken into consideration.

## Other improved AND circuits

We shall now consider a number of ways of improving AND circuits by looking at the circuit as a whole, rather than optimizing parts of the circuit which will act as good relays in series. It should be kept in mind that many of the tech-

*Figure* 3a **Circuit reliability as a function of input probability.** *Conjunction*



niques for improving the reliability will apply to OR circuits with the appropriate dualization.

(1) The simplest redundant network which will function as an AND circuit consists of $m$ basic AND circuits ($n$ relays in series) connected in parallel. This is somewhat analogous to the "system-standby" method of Moskowitz and McLean[3], where we switch to the redundant elements when required.

It is not obvious *a priori* whether there is an optimal $m$. It is obvious that the reliability cannot be made arbitrarily high. In fact, as $m \rightarrow \infty$, we shall see that the reliability of this circuit converges to 0. It is:

$$R_m{}^{(p)} = \sum_{k=0}^{n-1} \binom{n}{k} p^k q^{n-k} [1 - a^k c^{n-k}]^m + p^n [1 - (1 - a^n)^m] \qquad (8)$$

$$= \sum_{k=0}^{m} \binom{m}{k} [pa^k + (1-p)c^k]^n (-1)^k + p^n [1 - 2(1 - a^n)^m] \ .$$

While the above (nonlinear) sum cannot be expressed in closed form, the following bounds on $R_m{}^{(p)}$ are helpful in studying the variation of $R_m{}^{(p)}$ with $m$.

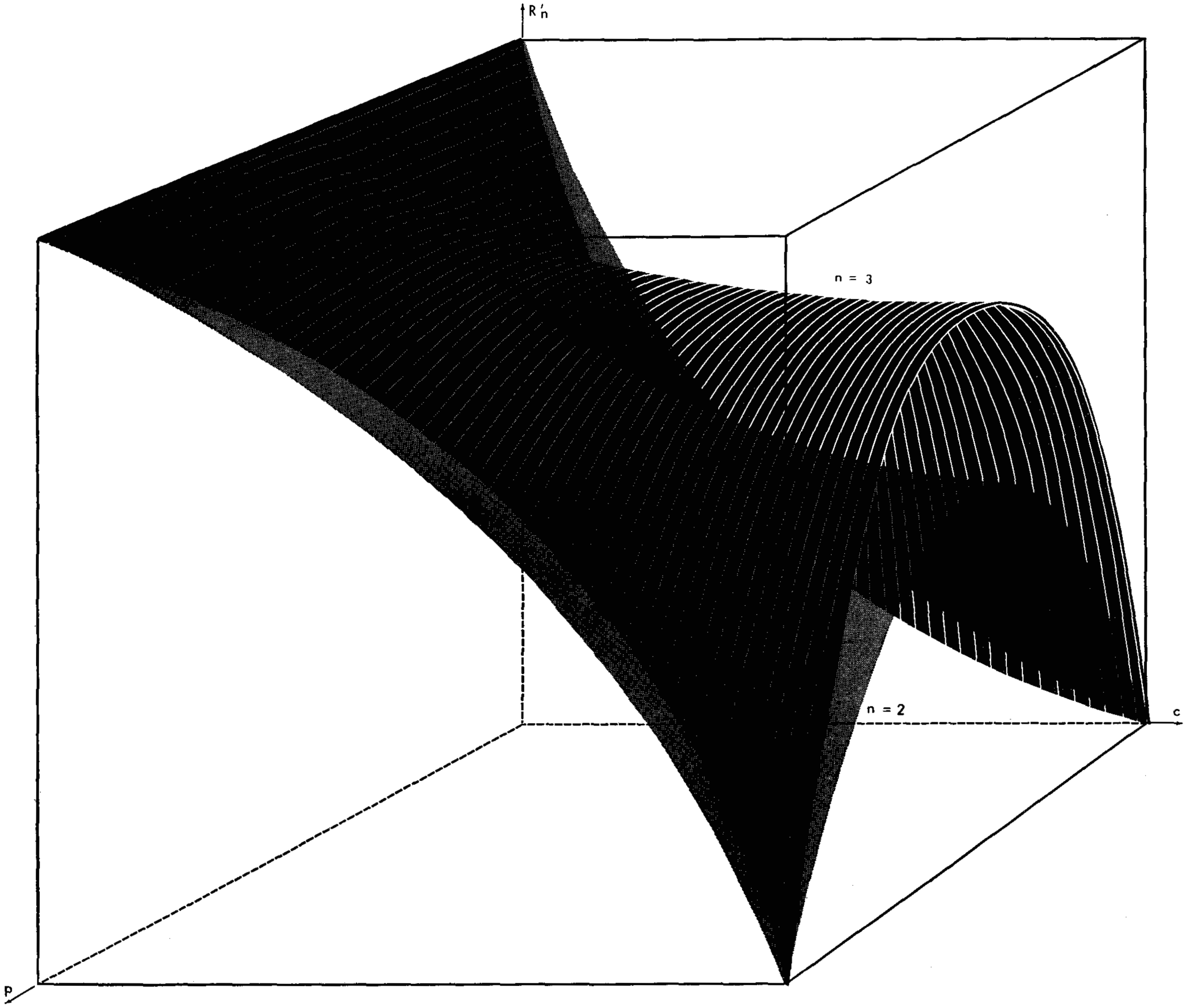
**Theorem 4.** *If* $p > c$, *and* $a = 1 - c$ *and* $c < 1/10$, *then*

$$1 - p^n \left\{ \tfrac{1}{2} \left[ \left( 1 + \frac{c}{p} \right)^n - 1 \right] [(1 + a^n)^m - (1 - a^n)^m] + (1 - a^n)^m \right\} \leq R_m{}^{(p)} \leq 1 \ .$$

*Proof:* See Appendix.

Unfortunately, this lower bound decreases with $m$, and

*Figure* 3*b* **Circuit reliability as a function of input probability.** *Disjunction*



is, therefore, not very informative as it stands. From Eq. (8), it is evident that if $c\neq 0$, $p\neq 1$, then $\lim_{m\to\infty} R_m{}^{(p)}=p^n$. From the above lower bound, however, it is possible to determine how $m$ must increase as a function of $n$ so that $\lim_{m\to\infty} R_m{}^{(p)}=1$.

**Theorem 5.** *If* $c<-p\ln p$ *and*

$$\exp[nc(1-\sqrt{c})]<m<\exp[nc(1-c)]\ ,\ \textit{then}\ \lim_{m\to\infty} R_m{}^{(p)}=1.$$

*Proof:* See Appendix.

Of course, for a given $n$, there is a maximum reliability less than 1, and the $m$ which achieves this maximum is given by the hypothesis of Theorem 5. The result is meaningful only for $n>1/c$, for example when very poor relays are available or if $n$ is unusually large. Except in such special cases, paralleling is not a good method of improving an AND circuit.

It may be mentioned in passing that placing $m$ basic AND circuits in series is an equally poor method. The reliability is:

$$R_m{}^{(s)}=\sum_{k=0}^{n-1}\binom{n}{k}p^kq^{n-k}(1-a^{mk}c^{m(n-k)})+p^na^{mn}$$

$$=1-[pa^m+qc^m]^n-p^n(1-2a^{mn})\ .$$

Clearly, $\lim_{m\to\infty} R_m{}^{(s)}=1-p^n$, which means that this circuit becomes like an open circuit as $m$ increases.

(2) We shall now investigate the simplest series-parallel net, consisting of $m'$ parallel lines, each a series of $m$ basic AND circuits. It is easy to see that such a circuit, with $m=m'=2$ is "single-error preventing", in the sense that failure of any single relay will not cause malfunction of the circuit, no matter which of the $2^n$ inputs occur. It can be shown that, for $n=2$, there is no "single-error preventing" circuit having fewer than eight relays. Generally, such an $(m+1)\times(m+1)$ series-parallel circuit is "$m$-tuple error preventing". Since the number of relays increases as $nmm'$, it is not obvious whether errors can be controlled in this way. In general, for instance if $m=m'$, it is no better than an open circuit. The reliability of this circuit is given by

$$R_m{}^{(sp)}=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}[1-c^{m(n-k)}a^{mk}]^{m'}$$

$$+p^k[1-2(1-a^{mn})^{m'}]\ .$$

Are there any conditions, specifying a relation between $m'$ and $m$, which make error control possible? The answer is in:

**Theorem 6.** *If* $c<e^{-n}$ *and* $m'=Ae^{mn}$ *where* $A$ *is a constant, then* $\lim_{m\to\infty} R_m{}^{(sp)}=1$.

*Proof:* See Appendix.

From the proof, it is seen that the larger the $A$, the larger the reliability. As an example, let $n=2$, $p=\frac{1}{2}$, $m=3$, $c=e^{-8}\doteq 0.00034$ and $Ae^{mn}=400$. The network will have a total of 2400 relays, with a probability of error of about $1.17\times 10^{-10}$.

(3) We now extend our investigation by connecting all the parallel elements in the above series-parallel network, as illustrated for the case of $n=2$ in Fig. 4 below. The circles and the squares are used to distinguish between contacts associated with the two different inputs. We shall call this a lattice network.

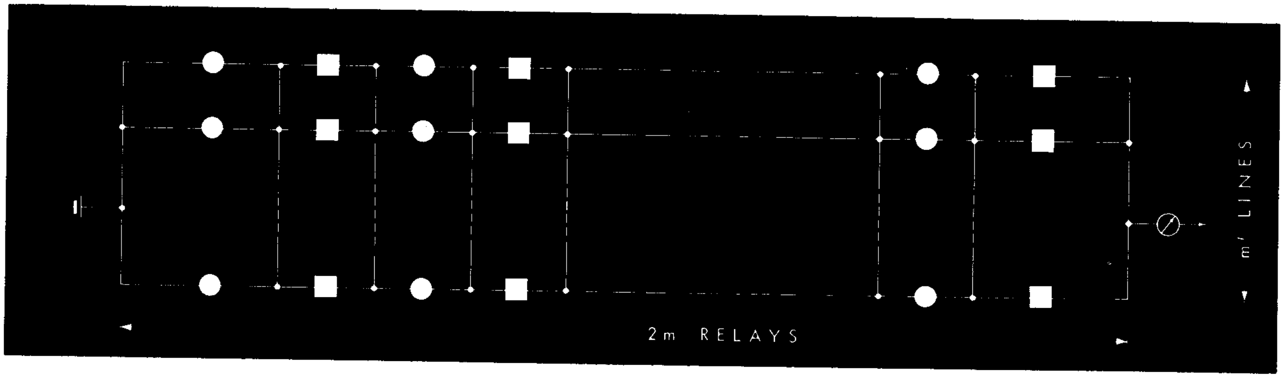
In general, there will be $m'$ parallel lines. Each line will contain $m$ basic AND circuits, or $mn$ relays in series. The reliability of this circuit is:

$$R_m{}^{(l)}=\sum_{k=0}^{n-1}\binom{n}{k}p^kq^{n-k}$$

$$\times\{1-[1-(1-c)^{m'}]^{m(n-k)}[1-(1-a)^{m'}]^{mk}\}$$

$$+p^n[1-(1-a)^{m'}]^{mn}\ .$$

*Figure 4* **Lattice network.**



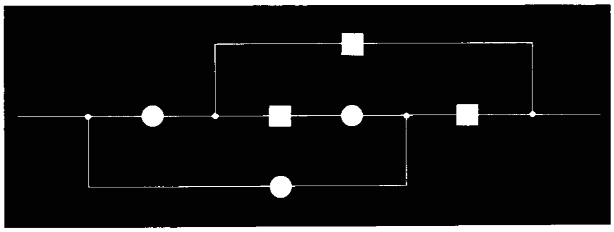


*Figure 5a* **Redundant AND circuit with loop.**

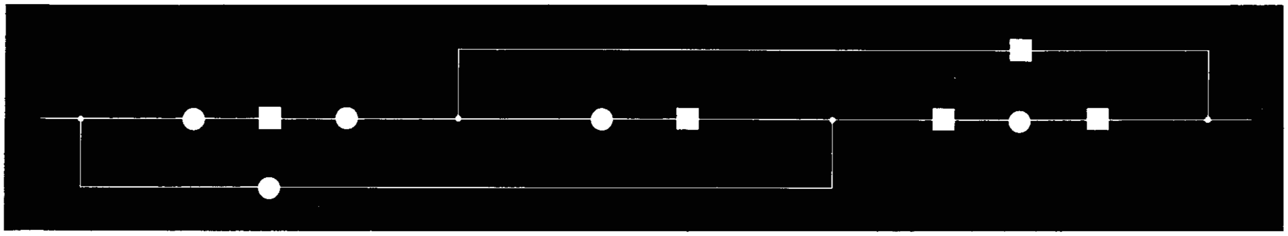


*Figure 5b* **Single-error preventing AND circuit.**

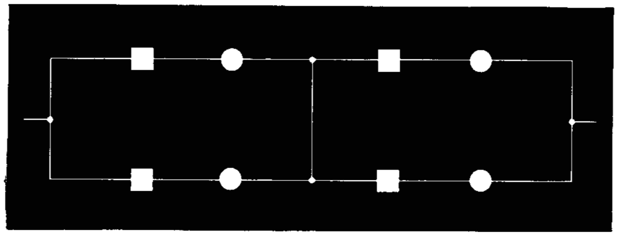


*Figure 6a* **Series-parallel redundant AND circuit.**

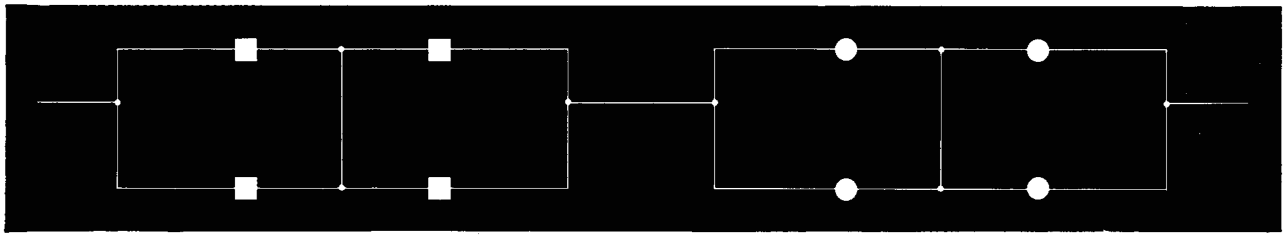


*Figure 6b* **Lattice AND circuit.**

We inquire again about special conditions under which error control is possible. While the circuit of Theorem 6 required enormous redundancies, in the form of $Ame^{mn}$, we shall see that this is a far better circuit in this regard. This is stated more precisely in the following basic result:

**Theorem 7.** *If* $c<1/e$ *and* $m'=B\ln m$, *where B is a constant, then* $\lim_{m\to\infty} R_m^{(l)}=1$.

*Proof:* See Appendix.

(4) We shall now examine some special circuits which act as two-way AND circuits, and show that the lattice network of Fig. 4 is a very good network. Consider first the 6-relay circuit of Fig. 5a. Its reliability is

$$R=q^2\{1-[2c^2(1-c)^4+0(c^3)]\}+p^2\{1-[3(1-a)^2a^4+0(c^3)]\}$$
$$+2pq\{1-[2c(1-c)^2a^3+3c^2(1-c)a^3$$
$$+4c(1-c)^2a^2(1-a)+0(c^3)]\} \;,$$

where $[0(x^k)/x^k]\to 1$ as $x\to 0$. If $c=1-a=2^{-10}$ and $p=\frac{1}{2}$, then $R=\frac{1}{4}[4 - 4\cdot 2^{-10}+19\cdot 2^{-20}+0(2^{-30})]$. Observe that this circuit will prevent single errors when both coils are energized or unenergized but not if one coil is energized while the other is not. The simplest circuit of this type which is single-error preventing for every input configuration is shown in Fig. 5b. Its reliability is

$$R=q^2[1-3c^4(1-c)^4-0(c^5)]+2pq[1-3c^2(1-c)^2a^4-0(c^3)]$$
$$+p^2[1-15(1-a)^2a^2(1-c)^4-0(c^3)]\doteq\frac{1}{4}[4-21c^2+0(c^3)]$$

for the above numerical values. This is, of course, much better than the network of Fig. 5a, but requires four relays more.

By comparison, consider two straightforward series-parallel nets illustrated by Figs. 6a and 6b. The second circuit is a special lattice network, as in Fig. 4, with $m=2$. The reliabilities corresponding to these circuits, evaluated with the above numbers, are:

$$R_a = q^2[1-0(c^4)]+2pq[1-4c^2(1-c)^2a^4+0(c^3)]$$
$$+p^2[1-8c^2(1-c)^2a^4+0(c^3)] \doteq \tfrac{1}{4}[4-16c^2+0(c^3)]$$

$$R_b = q^2[1-0(c^4)]+2pq[1-4c^2(1-c)^2a^4-0(c^3)]$$
$$+p^2[1-4c^2(1-c)^2a^4+0(c^3)] \doteq \tfrac{1}{4}[4-12c^2+0(c^3)] \ .$$

The lattice network is clearly the better. A comparison of Fig. 5a with 6a shows an increase in reliability by about $12c^2/4$ at the expense of only two additional relays. In contrast, a change from Fig. 6a to Fig. 5b, gives an increase in reliability of only $5c^2/4$, also at the expense of two additional relays.

A final special case which might be expected to work as an improved AND circuit is based on putting a basic AND circuit in parallel with a circuit which is equivalent to it by DeMorgan's theorem. That is, this circuit behaves according to the logical function $(x \cdot y)+(\overline{\bar{x}+\bar{y}})=z$, where $x$ and $y$ are Boolean variables (0 or 1), $\cdot$ denotes Boolean multiplication (AND) and $+$ denotes Boolean addition (OR); the two terms in parentheses denote the two parallel lines as shown in Fig. 7.

Assuming the relay labelled $L$ to be perfect, the reliability of this circuit is given by:

$$R = 1-[pa+qc]^n-[p(1-\bar{a})+q(1-\bar{c})]^n$$
$$+[pa(1-\bar{a})+(1-p)c(1-\bar{c})]^n+(ap)^n \ ,$$

where $\bar{a}$ and $\bar{c}$ denote the values of $a$ and $c$ for the normally closed relays. If $\bar{a}=c$ and $\bar{c}=a=1-c$, then

$$R = 1-2[p(1-c)+qc]^n+[p(1-c)^2+qc^2]^n+[p(1-c)]^n \ .$$

It is possible to show that this circuit is more reliable than a plain two-relay series circuit provided that $n \leq 3$ and $c > 2p/(7p-1)$. While this result is due, in part, to the looseness of the bounds on $n$ and $c$, which are difficult to improve because of the nonlinearity of the problem, it does point to the conclusion that this is not a good way of making an AND circuit except in very special cases (e.g., $p > 1/5$, very bad relays except for $L$).

(5) Another possibility for improving the reliability of an AND circuit consists of "composing" a series-parallel network, like that of Fig. 7, $m$ times, with the basic elements of the circuit being the simplest series AND circuits instead of the relays themselves. Let $r$ be the probability that an AND circuit is closed. Then, the probability that the entire circuit is closed is

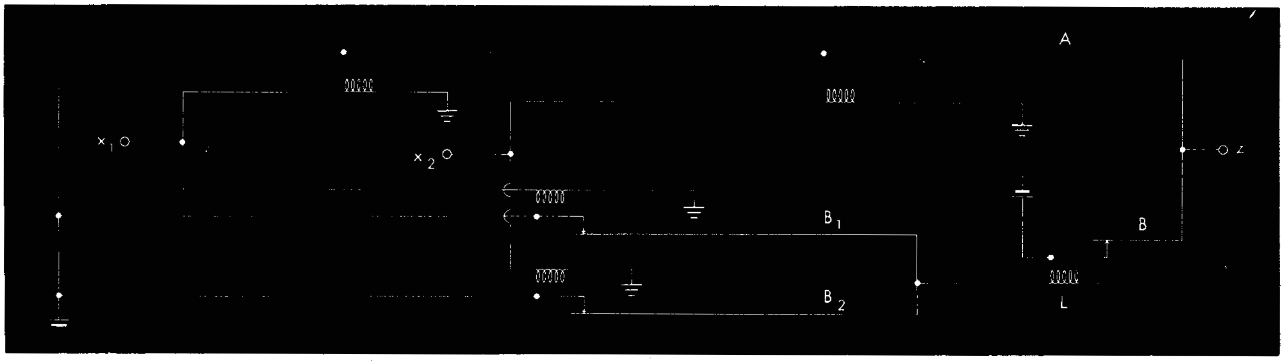
$$h(r) = \sum_{k=0}^{N} A_k r^k (1-r)^{N-k} \ ,$$

where $A_k$ is the number of ways in which the entire network of $N$ relays can be closed when exactly $k$ relays are closed and $N-k$ are open. Consider the network of Fig. 7, with an $n$-variable series AND circuit replacing each relay in that figure, and suppose the input is $11 \cdots 11$. Let $r=a^n$. If the new circuit is better than the plain series circuit, then $h(a^n)$ must exceed $a^n$, and $h[h[h \cdots [h(a^n)]].\ ] \cdots = h^{(m)}(a^n)$ must converge to 1 as $m \rightarrow \infty$ very rapidly. When the input is not all 1's, then the probability that the network is closed is at most $a^{n-1}c$. To function correctly as an AND circuit, it should, however, be open, and we would expect that $h(a^{n-1}c) < a^{n-1}c$, with $\lim_{m\rightarrow\infty} h^{(m)}(a^{n-1}c)=0$. Since $h(r)$ is a monotonically increasing function of $r$, $h(r) \leq h(a^{n-1}c)$ for all $r \leq a^{n-1}c$. Hence, we seek the smallest $m$ such that $h^{(m)}(a^n) > 1-\epsilon$ and $h^{(m)}(a^{n-1}c) < \epsilon$ for an arbitrary, positive $\epsilon$. With the circuit of Fig. 7, it is, of course, well known that such an $m$ exists only if $a^n > 0.38$ and $a^{n-1}c < 0.38$. Generally, improvement is possible provided that $a \neq c$. It is easily seen, however, that this is not a very economical way of increasing redundancy because the number of required relays increases as $n4^m$, while $h^{(m)}(1-\delta) = 1-\tfrac{1}{2}(2\delta)^{2m}$. That is, to achieve $\epsilon = 2^{-61}$, with $a^n = (1-2^{-10})^{32} \doteq 1-2^{-5} = 1-\delta$, $m$ should be 4 and $2^{13}$ relays would be required. Of course,

$$h^{(4)}(r) \leq \tfrac{1}{4}(4r)^{16} \leq \tfrac{1}{4}(4a^{n-1}c)^{16} \doteq \tfrac{1}{4}(4 \cdot 2^{-10})^{16} \doteq 2^{-126} \ .$$

Nevertheless, this technique of improving reliability, which is readily adapted to circuits other than AND circuits, is very useful when it is desired to control the probability of error for each input.

*Figure 7* **Redundant AND circuit based on De Morgan's theorem.**



## Improved OR, EXCLUSIVE-OR, and general circuits

Recalling the duality between OR and AND circuits and the results of the previous section, we might expect that the dual of the lattice network shown in Fig. 4 would be a good way of building an OR circuit. This circuit is shown in Fig. 8. Its reliability is:

$$R_m{}'^{(l)}=\sum_{k=1}^{n}\binom{n}{k}p^kq^{n-k}[1-(1-a)^{nk}(1-c)^{m(n-k)}]^{m'}+q^n\{1-[1-(1-c)^{mn}]^{m'}\}$$

$$\geq\sum_{k=1}^{n}\binom{n}{k}p^kq^{n-k}\exp\left\{-\frac{m'(1-a)^{mk}(1-c)^{m(n-k)}}{1-(1-a)^{mk}(1-c)^{m(n-k)}}\right\}+q^n\{1-\exp[-m'(1-c)^{mn}]\}\ .$$

Since we are dealing with normally open relays, we shall assume that $a>c$. It follows that $(1-a)^m(1-c)^{m(n-1)}>(1-a)^{mn}$, and that $(1-a)^{mk}(1-c)^{m(n-k)}\leq(1-a)^m(1-c)^{m(n-1)}$ for $k=1,\cdot\cdot\ ,n$. Therefore,

$$R_m{}'^{(l)}\geq\exp\left\{\frac{-m'(1-a)^m(1-c)^{m(n-1)}}{1-c}\right\}\sum_{k=1}^{n}\binom{n}{k}p^kq^{n-k}+q^n\left\{1-\exp\left[-m'e^{\frac{-mnc}{1-c}}\right]\right\}\ .$$

We now seek a relation between $m'$ and $m$ so that, for a sufficiently large $m$, the reliability can be made as close to 1 as desired.

**Theorem 8.** *If* $m'=Ae^{mn\sqrt{c}}$, *where* $A$ *is a constant, and* $n\sqrt{c}<1$, *then* $\lim_{m\to\infty}R_m{}'^{(l)}=1$.

*Proof:* We continue the computation of the expression for $R_m{}'^{(l)}$ which was started above by substituting for $m'$, and setting $c=1-a$.

$$R_m{}'^{(l)}\geq(1-q^n)\exp\left\{\frac{-Ae^{mn\sqrt{c}}c^m\ e^{-mc(n-1)}}{1-c}\right\}+q^n\left\{1-\exp\left[-Ae^{mn\sqrt{c}}\ e^{\frac{-mnc}{1-c}}\right]\right\}$$

$$=(1-q^n)\exp\left\{-\frac{A}{1-c}\ e^{m(n\sqrt{c}+\ln c-c(n-1))}\right\}+q^n\left\{1-\exp\left[-Ae^{mn\left(\sqrt{c}-\frac{c}{1-c}\right)}\right]\right\}\ . \tag{9}$$

From the condition $n\sqrt{c}<1$, and the fact that $n\geq2$, it follows that $c<\frac{1}{4}$, so that $\ln c<-1.39$. Therefore, $n\sqrt{c}-c(n-1)+\ln c<n\sqrt{c}+\ln c<1-1.39<0$. Hence, as $m\to\infty$, the argument of the first exp { } decreases to 0, and the first term of Eq. (9) converges to $1-q^n$. Next, observe that $\sqrt{c}-(c/1-c)>0$ because $(1-c)^2>c$ as a result of $c<\frac{1}{4}$. This causes the argument of the second exp { } in Eq. (9) to diverge as $m\to\infty$, so that the second term of Eq. (9) converges to $q^n\{1-0\}$. Hence, $R_m{}'^{(l)}\to1-q^n+q^n=1$.

The leading term in the expression for the probability of error is

$$Q=q^n\exp[-Ae^{mn(\sqrt{c}-c)}]\doteq q^n\exp[-Ae^{mn\sqrt{c}}]\ .$$

The total number of relays is $nmAe^{mn\sqrt{c}}$. Hence $A$ and $m$ should be chosen so that the first expression is as small as desired

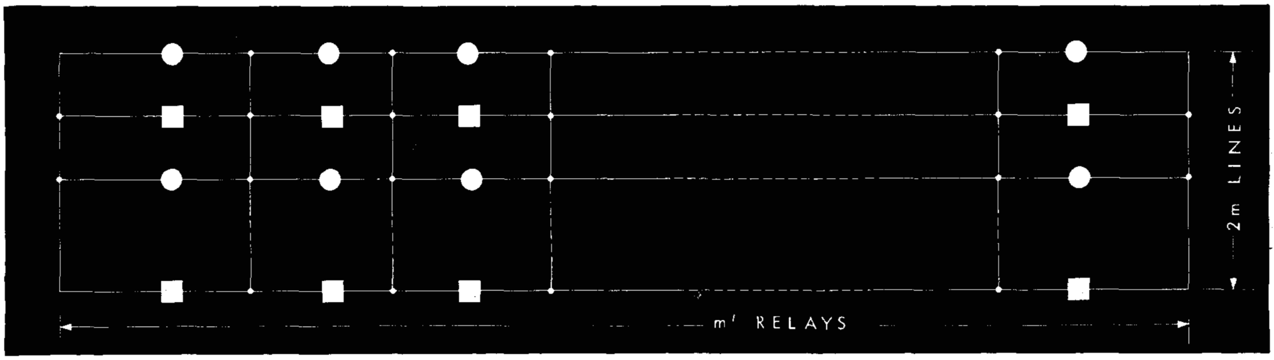


*Figure 8* **Lattice network as an OR circuit.**

(say $\epsilon$) and the second one as small as possible. The number of relays to obtain a given $\epsilon$ can be approximated by $-mn \ln (\epsilon/q^n)$; $m$ must be much larger than $\ln A$ for $Q$ to be a good approximation to the error probability.

*Example:* Let $n=2$, $q=\frac{1}{2}$, $c=2^{-10}$ and $\epsilon \leq e^{-21}$. If we take $A=7.4$ and $m=16$, we shall require a total of $2\times16\times7.4 \times e^{16\cdot2\cdot2^{-5}}=32\times55=1760$ relays. The reliability is

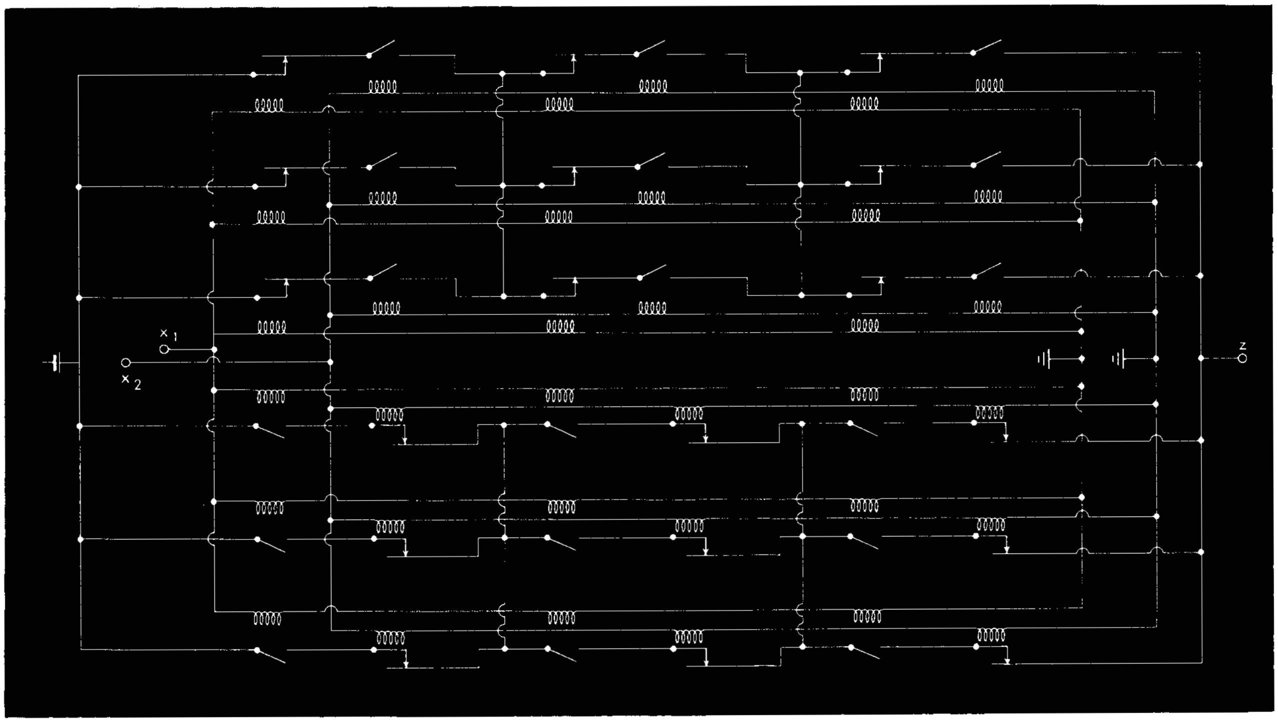
$$\tfrac{3}{4}\exp\{-7.4e^{-16\times6.93}\}+\tfrac{1}{4}\{1-\exp[-7.4e^{16\cdot2\cdot2^{-5}}]\} \doteq \tfrac{3}{4}[1-7.4e^{-112}]+\tfrac{1}{4}[1-e^{-20}] \doteq 1-\tfrac{1}{4}e^{-20} \ .$$

The techniques of improving AND circuits and OR circuits which have been described can now be used to design reliable circuits with arbitrarily specified logic, using the fact that any Boolean expression can be written as a sum of "maxterms" or product of "minterms" (see Ref. 3). That is, any circuit can be built with only AND and OR circuits and we know how to design these for reliable operation. As an example of what can be done, let us design an EXCLUSIVE-OR circuit which will fail at most once in $2^{57}$ operations on the average. Assume that $c=1-a=2^{-10}$, and for normally closed relays $a=1-c=2^{-10}$. Let $p=\frac{1}{2}$. The basic circuit has two inputs, coils, and the circuit should be closed if and only if one of the two input coils is energized and the other nonenergized. It is obvious how such a circuit can be built with two normally open and two normally closed relays. Basically, this circuit consists of two series AND circuits connected in parallel as an OR circuit. The circuit shown in Fig. 9 is an improved EXCLUSIVE-OR circuit. The reliability for this circuit is:

$$\begin{aligned}
R&=(1-p)^2\{1-[1-\big(1-(1-c)c\big)^m]^m\}^2+p^2\{1-[1-\big(1-c(1-c)\big)^m]^m\}^2\\
&\qquad+2p(1-p)\{1-[1-\big(1-(1-c)^{2m}\big)^m]\,[1-\big(1-(1-c^2)^m\big)^m]\}\\
&=[p^2+(1-p)^2]\{1-2[1-\big(1-c(1-c)\big)^m]^m+[1-\big(1-c(1-c)\big)^m]^{2m}\}\\
&\qquad\doteq\tfrac{1}{2}\{2-2m^m2^{-10m}+m^{2m}2^{-20m}+m^m2^{-20m}-m2^{-9m}+m^m2^{-10m}(1-m2^{-9m})\\
&\doteq 1-\tfrac{1}{2}[m^m2^{-10m}] \ .
\end{aligned}$$

The value of $m$ such that $\frac{1}{2}m^m2^{-10m}=2^{-57}$ is $m=8$. Thus, a total of $8\times8\times4=256$ relays are required for this circuit.

*Figure 9* **Reliable EXCLUSIVE-OR circuit.**



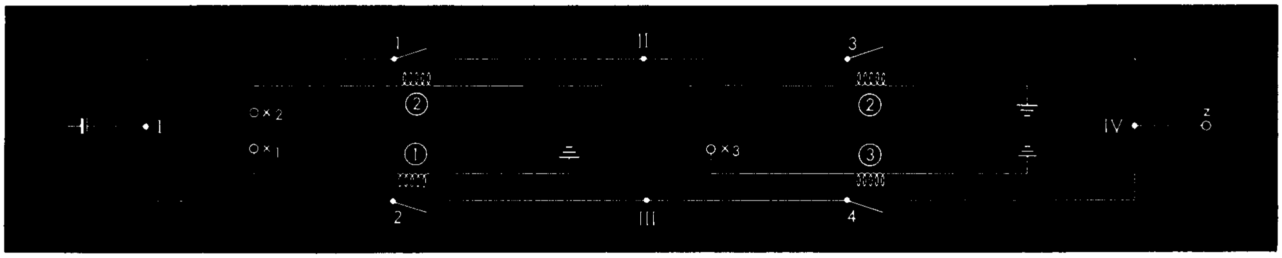


*Figure 10* **Labeling of a network.**

We shall conclude by showing how a digital computer may be used in the analysis, and possibly the synthesis, of reliable circuits with an arbitrarily specified logic. We start by assuming that an experienced circuit designer presents us with a large circuit which should be very reliable, and wishes to have its reliability computed by means of a computer program. Suppose that he presents this circuit by means of two incidence matrices, $D$ and $E$, and tells us the relay parameters $a$ and $c$ as well as the input distribution parameter $p$. Assuming that all relays operate perfectly, the behavior of the circuit is given by a sequence of $2^n$ bits, $z_0 \cdots z_{2^n-1}$, which specify whether the circuit is open (0) or closed (1) for the $2^n$ input configurations, which are enumerated in the order $x_1 \cdots x_n = 00 \cdots 00$; $00 \cdots 01$, $00 \cdots 10$, $00 \cdots 11$; $\cdots$; $11 \cdots 11$. Here $x_i = 1$ denotes that the $i^{\text{th}}$ input line is energized and $x_i = 0$ that it is not. The nodes, contacts and input lines of the circuit are arbitrarily numbered $1,\cdots,k,\cdots,N$; $1,\cdots,j,\cdots,m$; and $1,\cdots,i,\cdots,n$ respectively. The matrix element of $E_{n \times m}, e_{ij}$, is 1 if input line $i$ is coupled with contact $j$, and 0 if it is not; the matrix element of $D_{m \times N}, d_{kj}$, is 1 if contact $j$ is connected to node $k$ and 0 if not. This is best understood by examining an example, the circuit shown in Fig. 10, with $N = 4$ nodes (Roman numerals), $n = 3$ input lines (circled numbers) and $m = 4$ contacts (arabic numerals).

*Contact Number*

$$\begin{array}{cc|cccc|} & & 1 & 2 & 3 & 4 \\ \hline & \text{I} & 1 & 1 & 0 & 0 \\ \textit{Node Number} & \text{II} & 1 & 0 & 1 & 1 \\ & \text{III} & 0 & 1 & 0 & 1 \\ & \text{IV} & 0 & 0 & 1 & 1 \end{array} = D$$

*Contact Number*

$$\begin{array}{cc|cccc|} & & 1 & 2 & 3 & 4 \\ \hline & ① & 0 & 1 & 0 & 0 \\ \textit{Input Line Number} & ② & 1 & 0 & 1 & 0 \\ & ③ & 0 & 0 & 0 & 1 \end{array} = E$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| *Input Configurations:* | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
| $z_0 \cdots z_7$: | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |

Now let $r$, taking the values $0,1,\cdots,2^n-1$, denote the $r^{\text{th}}$ input configuration in the order of enumeration specified above. Let $s$, also taking values $0,1,\cdots,2^m-1$, denote the $s^{\text{th}}$ contact configuration, also lexicographically enumerated. Thus, $s = 0$ means that all contacts are open, and $s = 2^m - 1$ means that all $m$ contacts are closed. The first step in computing the circuit reliability, $R(D, E)$, is to compute $p_{rs}$, the conditional probability of contact configuration $s$, given input configuration $r$. Let $i_j$ be the value of $i$ for which $e_{ij} = 1$. Then, using the assumption that the contacts are independent, we have

$$p_{rs} = P[s = (s_1,\cdots,s_m)/r = (r_1,\cdots,r_n)] = \prod_{j=1}^{m} P(s_j/r_{i_j}) \ ,$$

where the $r_i$, $s_i$ are 0 or 1. Letting $p_r$ denote the probability of the $r^{\text{th}}$ input configuration, for example, $p_0 = (1-p)^n$, we see that $q_{rs} = p_r p_{rs}$ represents the joint probability of $r$ and $s$.

We must now enumerate all the contact configurations $r$ which, for any given $s$, give the correct circuit response. Thus, in a 2-relay AND circuit with both coils energized, there are three contact configurations which close the circuit. To this end, let $w_s$ be 1 if contact configuration $s$ closes the circuit, and 0 if it does not. Now observe that the circuit operates correctly only if the variables $z_r$ and $w_s$ are both 1 or both 0 for every $r$ and $s$. From this, it is easy to see that the reliability can be expressed as in Theorem 9.

**Theorem** 9.

$$R(D,E) = \sum_{r=0}^{2^n-1} \sum_{s=0}^{2^m-1} [z_r q_{rs} w_s + (1-z_r) q_{rs} (1-w_s)] \ .$$

A possible method for programming the exceedingly long computation according to the above formula consists of the following steps.

1. Compute the matrix $(q_{rs})$, which may require as many as $(m+1)2^{m+n}$ multiplications.

2. To obtain the vector **w**, having $w_s$ as the $s^{\text{th}}$ component, proceed as follows: Suppose, for example, that $s=2^m-1$ (all contacts closed). Starting with the first 1 of row 1 of $D$, scan down the column in which that 1 was found, and locate the other 1 in that column. Repeat this, using the row in which the latter 1 was located, instead of using the first row. Continue until either the last row is reached, which means that $w_s=1$, or until an entry is revisited and we begin to cycle through the matrix in a nonterminating loop. In the latter case, repeat the entire procedure by starting with the second instead of the first 1 of the first row, until either the last row is reached, or all possible paths through the matrix have been exhausted without reaching the last row, which means that $w_s=0$. To test configurations other than $s=2^m-1$, say the configuration $s=(010111\cdots1)$, simply replace columns 1 and 3 of $D$ by all 0's and repeat the above program. Determination of the vector **w** may take as many as roughly $2^m h^N$ searches through $D$, where $h$ is the maximum number of contacts incident on any node; each search requires at most $N^2$ decisions (whether an entry of $D$ is 0 or 1).

3. Assuming that the vector **z**, with $z_r$ as the $r^{\text{th}}$ component is given, the calculation of $R$ itself requires $2^{m+n+2}$ multiplications. Assuming, for purposes of illustration, that a multiplication (including fetching, storing and adding) takes about 100 microseconds, and that a decision takes about 10 microseconds, and letting $n=8$, $m=24$, $N=5$ and $h=4$, the calculation would take us no more than 3000 hours.

It is possible to devise clever programs which do not require as many as $h^N$ passes, on the average. It is faster to determine the presence of a closed path through a contact configuration than to determine that the circuit is open. The search is fastest for configurations where very few or where very many contacts are closed. Finally, the same paths through the circuit will be searched for several different configurations, and this would be avoided by the discovery of good algorithms for finding the shortest paths between two given points of a class of graphs. Combinatorial problems of this type present a challenge to programmers and analysts.

The synthesis problem, given $a$, $c$, $n$, $p$ and **z**, to find $N$, $m$ and matrices $D$ and $E$ such that $R(D,E)=1-\epsilon$, is even more difficult than the problem discussed above. One possibility is to start with some judiciously chosen matrices $D$ and $E$, vary some of their entries at random, observe the effect of this change on $R(D,E)$, and repeat to accumulate information about how $R$ varies as a function of $D$, $E$. Further changes in $D$ and $E$ will be increasingly based on such information. These steps would be much like those that clever engineers have taken in the synthesis of switching circuits before systematic methods were developed.

## Conclusion

We have shown how the redundancies which are inherent in the logic of switching circuit behavior can be used to advantage for the control of errors. As a measure of reliability, we used the probability that a given circuit functions as specified, averaged over all possible inputs. We investigated a number of redundant circuits which act reliably as AND, OR, and EXCLUSIVE-OR, and obtained formulas for bounds on the reliability. From these, we derived sufficient conditions on the topology of the circuits and its proportions for the possibility of error control. This means that we can make the reliability as close to 1 as we please by sufficiently increasing the redundancy of a circuit satisfying these conditions. Various circuits can be compared according to the redundancy required to achieve a specified reliability.

The first network which we studied consisted of $n$ "$m$-fold iterated $2\times2$ hammock networks" connected in series, designed to act like an $n$-way AND circuit. By choosing $m$ large enough, the reliability of this circuit can be made as large as desired. For example, with $n=32$, a value of $m=2$, involving a total of 512 relays, will suffice to produce a probability of error between $2^{-63}$ and $2^{-61}$. Contrast this with the Moore-Shannon technique of producing an AND circuit of the same reliability, in which at least 1440 relays are needed. This result is a consequence of our definition of reliability with all inputs assumed equiprobable and all errors equally important. Secondly, we showed that placing a number of basic AND circuits in parallel or series was not a good way of improving them. Neither was placing an AND circuit in parallel with a circuit equivalent to an AND by DeMorgan's Theorem. Next, we studied a network consisting of $m'$ parallel lines, each containing $m$ basic $n$-way AND circuits in series. It is an important fact that if $m'=Ae^{mn}$, where $A$ is a constant, then as $m$ increases without bound, the reliability converges to 1. Thus, if $n=2$, and each relay has a failure rate of about 0.00034, then a total of 2400 relays ($m=3$) gives an error probability of $1.17\times10^{-10}$. Among the circuits investigated, one resembling the above $m'\times m$ series-parallel net but with $mn$ lines connecting all neighboring parallel lines so as to form an $m'\times mn$ rectangular grid, seemed to be best. It was proved that, if $m'$ is proportional to $\log m$, then the reliability tends to 1 as $m$ increases. Thus, if $n=2$ and the failure rate of a relay is about 0.001, the unreliability of this circuit as an AND is about $e^{-21}$, requiring a total of 1760 relays ($m=16$). Similar results were obtained for OR circuits and EXCLUSIVE-OR circuits.

A method for improving "reliability" which is not based on averaging over all inputs was also studied and found to require enormous redundancies. We have also outlined an algorithm for computing the reliability of any switching circuit and have indicated how this might be implemented with a computer program.

The main contribution of this article is, however, in that it develops new analytical techniques, presents some novel inequalities of primarily mathematical interest, and deepens our understanding of the Moore-Shannon model for switching circuits.

The assumptions of this model represent the limitations on the applicability of our results to physical switching circuits. All switches are thus supposed to be independent, with no aging or catastrophic failures, and with failures occurring only in the contacts (no wire breaks or loose solder joints). Possibly relays, and perhaps cryotrons, meet these conditions to some extent. Since this model has, up to now, only investigated how to replace a single relay by a redundant network which should act reliably as a single relay, this study extends the scope of the model. There are, furthermore, no other theoretical results on the reliability of switching circuits comparable to those presented here.

If considering all the failures of a circuit equally important is objectionable, then the various error probabilities should be weighted according to their importance in computing the average error probability. The estimation of such weights and the determination of either physical components to which this model fits well, or a revised model more closely patterned after the physics of failing switches, are two outstanding problems for empirical study.

As in previous reliability studies, a remarkable feature common to all results is the enormous rate at which the redundancy required to produce a slight increase in reliability grows with the desired reliability. In order to obtain reliability at more reasonable cost, if possible even in principle, we seem to need a different method for organizing circuits than has been studied so far. Possibly, the organization of the nervous system may provide clues. When such a principle is proposed, the analytical tools developed in the meantime will be available to help us discover it.

**Acknowledgment**

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## Appendix 1: Proof to Theorem 2

To obtain a lower bound for $2^n Q_m=(a_m+c_m)^n-2a_m{}^n+1$, we shall first show that $(2-d)^{-2}[c(2-d)^2]^{2^m}$ is a lower bound for $c_m$. From Eq. (3), it follows that $c_{m+1}\leq c_m$, because $c_m-c_{m+1}=c_m[1-4c_m+4c_m{}^2-c_m{}^3]$, which is positive provided that $c_m<\frac{1}{4}$; thus, if $c<\frac{1}{4}$, then $c_1<c$, and the result follows. Now, from this we infer that $c_m\leq d$ for all $m$ so that $c_{m+1}\geq c_m{}^2(2-d)^2$. Observe that $c_1\geq c^2(2-d)^2$, $c_2\geq c_1{}^2(2-d)^2\geq c^4(2-d)^{4+2}$, $c_3\geq c_2{}^2(2-d)^2\geq c^8(2-d)^{8+4+2}$et cetera.
In general,

$$c_m\geq c^{2^m}(2-d)^{2^m+2^{m-1}+2^{m-2}+\dots+2}=c^{2^m}(2-d)^{2^{m+1}-2}=(2-d)^{-2}[c(2-d)^2]^{2^m} \quad ,$$

which will be abbreviated by $y$. We have thus also shown that $c_m$ decreases monotonically with $m$.

We shall now show that $1-\frac{1}{2}(2c)^{2^m}$ is a lower bound for $a_m$. Let $a_m=1-e_m$, with $e_0=c$. By Eq. (3), $a_{m+1}=(1-e_m)^2(1+e_m)^2$ $=(1-e_m{}^2)^2=1-e_m{}^2(2-e_m{}^2)$, so that $e_{m+1}=e_m{}^2(2-e_m{}^2)$. Hence, $e_{m+1}\leq 2e_m{}^2$ for all $m$. Consequently, $e_1\leq 2c^2$, $e_2\leq 2e_1{}^2\leq 2(2c^2)^2$, $e_3\leq 2c_2{}^2\leq 2^1 2^2 2^4 c^8$ etc., and $e_m\leq 2^{1+2+\dots+2^{m-1}}c^m=\frac{1}{2}(2c)^{2^m}$. Therefore, $a_{m+1}\geq 1-\frac{1}{2}(2c)^{2^m}$.

It is only necessary to obtain an upper bound for $a_m$, or a lower bound on $e_m$. Proceeding as in the first part of this proof, we note that $e_{m+1}\leq e_m$, and conclude that $e_{m+1}\geq c^{2^m}(2-d^2)^{1+2+4+\dots+2^{m-1}}=(2-d^2)^{-1}[(2-d^2)c]^{2^m}$. We shall abbreviate the latter quantity by $x$. It decreases towards 0 as $m$ increases.

Substituting these bounds into $Q_m$ we have: $2^n Q_m\geq[1-\frac{1}{2}(2c)^{2^m}+y]^n-2(1-x)^n+1\geq 1-n[\frac{1}{2}(2c)^{2^m}-y]-2(1-x)^n+1$ .

In order to establish a useful lower bound for $Q_m$, we must obtain as small an upper bound for $(1-x)^n$ as we can. To this end, we shall show that $(1-x)^n\leq 1-nx(\cosh n^2x^2-\sinh nx)$ .

To prove this, write $(1-x)^n\leq e^{-nx}=1-nx\left[1-\frac{nx}{2!}+\frac{(nx)^2}{3!}-\frac{(nx)^3}{4!}+\cdots\right]$ .

Observe that $\frac{(nx)^k}{(k+1)!}\leq\frac{(nx)^k}{k!}$ for $k=2,4,6,8,\dots$

and $\frac{(nx)^k}{(k+1)!}\geq\frac{(nx)^{2k}}{k!}$ for $k=1,3,5,7$, because $\frac{1}{k+1}\geq(nx)^k$ , provided that $0\leq x\leq 1/(ne)$ .

Hence,

$$\begin{aligned}(1-x)^n&\leq 1+nx\left[\frac{nx}{1!}+\frac{(nx)^3}{3!}+\cdots\right]-nx\left[1+\frac{(n^2x^2)^2}{2!}+\frac{(n^2x^2)^4}{4!}+\cdots\right]\\&=1+nx[\sinh nx-\cosh n^2x^2] \ .\end{aligned}$$

Thus,

$$2^nQ_m \geq 2-n[\tfrac{1}{2}(2c)^{2^m}-y]-2+nx(\cosh n^2x^2-\sinh nx) \ ,$$

so that

$$Q'_m = 2^{-n}[nx(\cosh n^2x^2-\sinh nx)-n/2(2c)^{2^m}+ny] \ . \qquad \text{QED.}$$

## Appendix 2: Proof to Theorem 3

We shall first derive an upper bound for $c_m$. Starting from Eq. (3), we write: $c_{m+1} \leq 4c_m{}^2$; hence, $c_1 \leq 4c^2$, $c_2 = 4c_1{}^2 \leq 4 \cdot 4^2c^4$ $c_3 \leq 4 \cdot 4^2 \cdot 4^4c^8$, et cetera, and generally, $c_m \leq 4^{1+2+4+\ldots+2^{m-1}}c^{2^m} = \frac{1}{4}(4c)^{2^m}$.

Substituting the upper and lower bounds for $a_m$ which were derived in the proof of the Theorem 2, we have:

$$2^nQ_m \leq [(1-x+\tfrac{1}{4}(4c)^{2^m}]^n-2(1-\tfrac{1}{2}(2c)^{2^m})^n+1 \leq 1-nz(\cosh n^2z^2-\sinh nz)-2(1-n/2(2c)^{2^m}+1$$

$$= n[(2c)^{2^m}+z \sinh nz-z \cosh n^2z^2] \ ,$$

where $z = \dfrac{[(2-d)^2c]^{2^m}}{(2-d^2)}-\frac{1}{4}(4c)^{2^m}$ . QED.

A somewhat higher, but more informative, upper bound is obtained by using:

$$1-\tfrac{1}{2}(2c)^{2^m} \leq a_m \leq 1 \text{ and } (1-\epsilon)^n \leq 1+\epsilon\sum_{k=1}^{n}\binom{n}{k} \leq 1+2^n\epsilon \ , \text{ for } \epsilon^k \leq \epsilon \ .$$

Thus,

$$2^nQ_m \leq [1+\tfrac{1}{4}(4c)^{2^m}]^n-2[1-\tfrac{1}{2}(2c)^{2^m}]+1 \leq 1+2^{n-2}(4c)^{2^m}-2(1-n/2(2c)^{2^m})+1$$

$$= (2c)^{2^m}(2^{2^m+n-2}+n) = (4c)^{2^m}2^n(\tfrac{1}{4}+n2^{-2^m}) \leq (4c)^{2^m}2^n\tfrac{1}{2}(n+\tfrac{1}{2}) \leq (4c)^{2^m}n2^n \ .$$

## Appendix 3: Proof to Theorem 4

First rewrite the expression for $R_m{}^{(p)}$ as:

$$R_m{}^{(p)} = 1+\sum_{k \text{ even}, \geq 2}\binom{m}{k}[pa^k+(1-p)c^k]^n-\sum_{k \text{ odd}}\binom{m}{k}[pa^k+(1-p)c^k]^n+p^n[1-2(1-a^n)^m] \ .$$

Next, observe that $p \geq c$ implies that $(1-p) \leq a$, so that $(1-p)c \leq c/p(pa)$, and $(1-p)c^k \leq c/p \, pa^k$, $k = 1,2,3,\cdots$. Applying this inequality for odd $k$, and setting $\epsilon = c/p$, we have:

$$R_m{}^{(p)} \geq 1+\sum_{k \text{ even}, \geq 2}\binom{m}{k}[pa^k]^n-\sum_{k \text{ odd}}\binom{m}{k}[(1+\epsilon)pa^k]^n+p^n[1-2(1-a^n)^m]$$

$$= 1+p^n\left\{\sum_{k=0}^{m}\binom{m}{k}(a^n)^k(-1)^k-[(1+\epsilon)^n-1]\sum_{k \text{ odd}}\binom{m}{k}(a^n)^k-2(1-a^n)^m\right\}$$

$$= 1+p^n\left\{(1-a^n)^m-\tfrac{1}{2}[(1+\epsilon)^n-1]\left[\sum_{k=0}^{m}\binom{m}{k}(a^n)^k-\sum_{k=0}^{m}\binom{m}{k}(a^n)^k(-1)^k\right]-2(1-a^n)^m\right\}$$

$$= 1-p^n\{\tfrac{1}{2}[(1+\epsilon)^n-1][(1+a^n)^m-(1-a^n)^m]+(1-a^n)^m\} \ .$$

For $c$ very small and $m$, $n$ very large, this expression can be approximated by

$$1-p^n\left(\frac{nc}{p} \sinh me^{-nc}+e^{-me^{-nc}}\right) \ .$$

In a similar manner, an upper bound can be derived for $R_m{}^{(p)}$, but, unfortunately, it exceeds 1 for large $m$, so that $R_m{}^{(p)} \leq 1$ is the strongest inequality which can presently be stated.

## Appendix 4: Proof to Theorem 5

We shall first prove an inequality which will be frequently used in the sequel.

*Lemma: If* $0 \leq x \leq 1$, $n \geq 1$, *then* $e^{-n\frac{x}{1-x}} \leq (1-x)^n \leq e^{-nx}$

The right-hand inequality is well known. To prove the left-hand side, let $u=(1-x)^n$.

Then $\ln u = n \ln (1-x) = -n\int_0^x \frac{d\xi}{1-\xi} \geq -n\frac{x}{1-x}$ . QED.

Let $Q_m{}^{(p)} = p^n\left\{\frac{1}{2}\left[\left(1+\frac{c}{p}\right)^n - 1\right][(1+a^n)^m-(1-a^n)^m]+(1-a^n)^m\right\}$

From the inequalities

$$\left(1+\frac{c}{p}\right)^n \leq e^{-me^{-\frac{nc}{1-c}}},\ (1+a^n)^m \leq e^{ma^n} \leq e^{me^{-nc}},\ (1-a^n)^m \leq e^{-ma^n} \leq e^{-me^{-\frac{nc}{1-c}}},\ \text{and}\ (1-a^n)^m \geq e^{-\frac{ma^n}{1-a^n}} \geq e^{-\frac{me^{-nc}}{1-e^{-nc}}},$$

it follows that:

$$Q_m{}^{(p)} \leq e^{n \ln p}\left\{\frac{1}{2}\left[e^{nc/p}-1\right]\left[e^{me^{-nc}}-e^{-\frac{mc^{-nc}}{1-e^{-nc}}}\right]+e^{-me^{-\frac{nc}{1-c}}}\right\}$$

$$=e^{n \ln p}\left\{\frac{1}{2}\left[\exp\left(\frac{nc}{p}+me^{-nc}\right)-\exp\left(\frac{nc}{p}-\frac{me^{-nc}}{1-e^{-nc}}\right)-\exp\,(me^{-nc})+\exp\left(-\frac{me^{-nc}}{1-e^{-nc}}\right)\right]+\exp\left(-me^{-\frac{nc}{1-c}}\right)\right\}\ .$$

From the hypothesis it follows that $e^{-nc^{3/2}} < e^{-nc} < e^{-nc^2}$, so that:

$$Q_m{}^{(p)} \leq \frac{1}{2}\left[\exp\left(n \ln p+\frac{nc}{p}+e^{-nc^2}\right)-\exp\left(n \ln p+\frac{nc}{p}-\frac{e^{-nc^2}}{1-e^{-nc}}\right)\right.$$

$$\left.-\exp\left(e^{-nc^{3/2}}+n \ln p\right)+\exp\left(-e^{-\frac{nc^{3/2}}{1-e^{-nc}}}+n \ln p\right)\right]+\exp\left(-e^{-\frac{nc^{3/2}}{1-c}}+n \ln p\right)\ .$$

From $c+p \ln p<0$ it follows that $n[\ln p+(c/p)]$ decreases linearly as $n$ increases, from $2/p(p \ln p+c)$, a large negative quantity, to $-\infty$, while $e^{-nc^2}$ decreases from about 1 to 0. Hence, the first and second terms in the above expression converge to 0 as $m\rightarrow\infty$. The third, fourth and fifth terms clearly also converge to 0, because $\ln p<0$. Hence $\lim\limits_{m\rightarrow\infty} Q_m{}^{(p)}=0$, and the result follows.

## Appendix 5: Proof to Theorem 6

$$R_m{}^{(s,p)} = \sum_{k=0}^{n-1}\binom{n}{k}p^k(1-p)^{n-k}[1-c^{m(n-k)}a^{mk}]^{Ae^{mn}}+p^n[1-(1-a^{mn})^{Ae^{mn}}]c^{m(n-k)}a^{mk} \leq c^m a^{m(n-1)} \leq a$$

$$\therefore [1-c^{m(n-k)}a^{mk}]^{Ae^{mn}} \geq \exp\left[\frac{-Ae^{mn}c^m a^{m(n-1)}}{1-a}\right]$$

$$\geq \exp\left[-\frac{A}{1-a}e^{mn}e^{m \ln c}e^{-mc(n-1)}\right] \quad \text{because} \quad a^{m(n-1)} \leq e^{-mc(n-1)}$$

If $c<e^{-n}$, then $n+\ln c-c(n-1)<-c(n-1)<0$, so that each bracketed term in the above summation converges to 1 as $m\rightarrow\infty$, making the sum converge to $1-p^n$. Now,

$$(1-a^{mn})^{Ae^{mn}} < \exp\,[-Ae^{mn}a^{mn}] < \exp\left[-Ae^{mn}e^{\frac{-mnc}{1-c}}\right]=\exp\left[-Ae^{\frac{mn(1-c)-mnc}{1-c}}\right]=\exp\left[-Ae^{\frac{mn(1-2c)}{1-c}}\right]\ .$$

Since $c<\frac{1}{2}$, this term converges to 0 as $m\rightarrow\infty$, and $R_m{}^{(sp)}\rightarrow 1-p^2+p^2=1$.

## Appendix 6: Proof of Theorem 7

$$R_m{}^{(l)} \geq \sum_{k=0}^{n-1} \binom{n}{k} p^k q^{n-k} \{1 - e^{-m(n-k)(1-c)^{m'}} e^{-mk(1-a)^{m'}}\} + p^n[1 - c^{m'}]^{mn}$$

$$\geq \sum_{k=0}^{n-1} \binom{n}{k} p^k q^{n-k} \left\{1 - \exp\{-m[(n-k)e^{\frac{-m'c}{1-c}} + kc^{m'}]\}\right\} + p^n[1 - mn(c^{m'})^{\alpha}]$$

$$\geq (1-p^n)[1 - \exp(-mnc^{m'})] + p^n[1 - mnc^{\alpha m'}]$$

$$= 1 - \exp(-mnc^{m'}) + p^n[1 - mnc^{\alpha m'} - 1 + \exp(-mnc^{m'})]$$

where $\alpha = \{\ln[1-(1-c^{m'})^{mn}] - \ln mn\}/\ln c^{m'}$

Setting $m' = B \ln m$, we have:

$$R_m{}^{(l)} \geq 1 - \exp(-nm^{1+A \ln c}) + p^n[e^{-nm^{1+B \ln c}} - nm^{1+\alpha B \ln c}]$$

Choose $B$ such that $A \ln c > -1$, so that both exponentials converge to 0 as $m \rightarrow \infty$. We shall now show that $\alpha B \ln c < -1$, so that the last term also converges to 0 and $\lim_{m \rightarrow \infty} R_m{}^{(l)} = 1$. Now,

$$\alpha \geq \frac{\ln[1 - e^{-nm^{1+B \ln c}}] - \ln n - \ln m}{\ln m^{A \ln c}}$$

so that

$$\alpha B \ln 1/c \geq \frac{\ln[1 - e^{-nm^{1+B \ln c}}] - \ln n - \ln m}{-\ln m} = 1 + \frac{\ln n}{\ln m} - \frac{\ln[1 - e^{-m^{1+B \ln c}}]}{\ln m} \geq 1$$

because the last term is positive.

Note that $m' \geq 1$. Hence, from $B < 1/\ln(1/c)$ it follows that $\ln m \geq \ln 1/c$ and $m \geq 1/c$. Hence, for relatively large $c$ this is a good method for improving reliability.

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